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THE JOURNAL

OF THE

Indian Mathematical Society

Vol. X.] FEBRUARY 1918 [No. 1

EDITED BY

M. T. NARANIENGAR, M. A.

Hony. Joint Secretary,

WITH THE CO-OPERATION OF

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A paper should contain a short and clear summary of the new results obtained and the relations in which they stand to results already known. It should be remembered that, at the present stage of mathematical research, hardly any paper is likely to be so completely original as to be independent of earlier work in the same direction; and that readers are often helped to appreciate the importance of a new investigation by seeing its connexion with more familiar results.

The principal results of a paper should, when possible, be enunciated separately and explicitly in the form of definite theorems.

The Journal is open to contributions from members as well as subscribers. The editors may also accept contributions from others.

Contributors will be supplied, if so desired, with extra copies of their contributions at net cost.

All contributions should be written legibly on one side only of the paper, and all diagrams should be given on separate slips.

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THE JOURNAL

OF THE

Indian Mathematical Society.

Vol. X.] FEBRUARY 1918. [No. 7.

PROGRESS REPORT.

The following gentlemen have been elected members of our Society :—

1. *Mr. A. M. Koreishi, M.A.*—Professor of Mathematics, M. A. O. College, Aligarh (U. P.)
2. *Mr. G. V. Seshiah, B.A.*—Mathematics Teacher, Athithota, Nellore.

2. The Committee record with great regret the sad and premature death of Mr. P. R. Krishnaswami, M.A., formerly of Pachayappa's College, Madras. Mr. Krishnaswami served as Assistant Secretary of our Society during 1915 and 1916, when his failing health forced him to resign his post. In him the Society has lost an earnest and enthusiastic member.

3. The following books have been received for the Library :—
   1. Calcutta University Calendar, for 1917, Parts II & III.
   2. Madras University Calendar, for 1917, Part I.

4. The Hon. Treasurer's Statement of Accounts for the year 1917 is given overleaf.

POONA,
31*st January* 1918.

D. D. KAPADIA,
Hon. Joint Secretary.

THE INDIAN MATHEMATICAL SOCIETY.

Statement of Accounts of the Society for the year 1917.

Receipts.	Rs.	A.	P.	*Expenditure.*	Rs.	A.	P.
To Balance from 1916	1,946	2	1	By ordinary working charges	176	14	8
„ Subscriptions from Members	1,870	0	0	„ Books and Journals	266	8	3
„ Subscriptions for Journal	236	0	0	„ Printing Journal	502	2	0
„ Interest on investments	70	4	8	„ Library	327	6	0
„ Miscellaneous receipts	147	12	6	„ Closing balance*	2,97	4	4
Total ...	4,270	3	3	Total ...	4,270	3	3

I have examined the Treasurers' books and vouchers as also the monthly statements submitted by the Secretary, the Assistant Secretary and the Assistant Librarian and declare the above accounts to be correct.

(Sd.) L. N. SUBRAMANIAN,

3rd February 1918. *Hon. Auditor.*

*

	Rs.	A.	P.
Madras Central Urban Bank	1,200	0	0
The Indian Bank	630	4	8
Treasurer	958	5	8
Assistant Secretary	136	8	6
Secretary	58	6	0
Assistant Librarian	4	0	0
Joint Editor	9	11	6
Total ...	2,997	4	4

MADRAS,
30*th January* 1918.

S. NARAYANA AIYAR,
Hon. Treasurer.

Infinite Series and Arithmetical Functions.

By Prof. F. Hallberg.

(*Continued from p.* 186, *Vol. IX.*)

III. Laguerre's series.

12. I now pass on to the interesting series

$$\sum_{n=1}^{\infty} f(n)\frac{x^n}{1-x^n} = \sum_{n=1}^{\infty} x^n \psi(n), \quad |x|<1 \quad \ldots \quad \ldots \quad (12.1)$$

where $\psi(n)$ is the "numerical integral" of $f(n)$

$$\psi(n) = \sum_{d:\, n} f(d), \quad \ldots \quad \ldots \quad \ldots \quad (12.11)$$

the sum being extended over all divisors d of n. The formula can easily be proved if the left side is developed in a double series.

We know then from the theory of numbers, that

$$f(n) = \sum_{d:\, n} \mu(d)\psi\left(\frac{n}{d}\right), \quad \ldots \quad \ldots \quad \ldots \quad (12.12)$$

a formula proved by R. Dedekind* and J. Liouville†.
(12.1) may therefore be written in two ways:

$$\sum_{n=1}^{\infty} f(n)\frac{x^n}{1-x^n} = \sum_{n=1}^{\infty} x^n \sum_{d:\, n} f(d), \quad \ldots \quad \ldots \quad (12.13)$$

$$\sum_{n=1}^{\infty} \frac{x^n}{1-x^n} \sum_{d:\, n} \mu(d)\psi\left(\frac{n}{d}\right) = \sum_{n=1}^{\infty} x^n \psi(n). \quad \ldots \quad \ldots \quad (12.14)$$

The functions $f(n)$, $\psi(n)$ may here be chosen arbitrarily, provided only the series converge.

13. *Some Special Cases.*

For $f(n)=1$ we obtain Lambert's series

$$L(x) = \sum_{n=1}^{\infty} \frac{x^n}{1-x^n} = \sum_{n=1}^{\infty} t(n)x^n, \quad |x|<1 \quad \ldots \quad \ldots \quad (13.1)$$

a series used by numerous writers in connection with the difficult theory of the asymptotic distribution of the primes.

---

* J. reine angew. Math. 54 (1857), p. 1.

† J. math. pures appl. (2) 2 (1857), p. 110.

On the other hand, knowing that

$$\sum_{n=1}^{\infty} \frac{x^n}{1-x^n} \sim \frac{1}{1-x} \log \left(\frac{1}{1-x} \right) \text{ [Cesàro]}, \quad \ldots \quad (13.11)$$

we may expect that, as $\sum_{h=1}^{n} t(h)$ is an increasing function of n,

$$\sum_{h=1}^{n} t(h) = n \log n + O(n) \quad \ldots \quad \ldots \quad (13.12)$$

In fact, Dirichlet has proved that

$$\sum_{h=1}^{n} t(h) = n \log n + (2C-1)n + O(\sqrt{n}) \quad \ldots \quad (13.13)$$

C being Euler's constant.

For $f(n) = n$, we have Euler's result

$$\sum_{n=1}^{\infty} \frac{nx^n}{1-x^n} = \sum_{n=1}^{\infty} x^n f(n) \quad , |x| < 1. \quad \ldots \quad (13.2)$$

where $f(n)$ denotes the sum of the divisors of n.

Thus

$$(1-x) \sum_{n=1}^{\infty} \frac{nx^{n-1}}{1+x+x^2+\ldots+x^{n-1}} = \frac{xf(1)+x^2f(2)+x^3f(3)+\cdots}{x+2x^2+3x^3+\ldots} \quad (13{\cdot}21)$$

Now Dirichlet has proved, that as $n \to \infty$,

$$\sum_{h=1}^{n} f(h) = \frac{\pi^2 n^3}{12} + O(n \log n), \quad \ldots (13.22)$$

and we see, because of a theorem of Cesàro, that the right side of (13.21) cannot oscillate between wider limits when $x \to 1$ than does

$$\frac{\frac{\pi^2 n^2}{12} + O(n \log n)}{\frac{1}{2} n(n+1)}, \quad \text{when } n \to \infty \ ,$$

which expression approaches $\frac{\pi^2}{6}$.

Hence we obtain

$$\sum_{n=1}^{\infty} \frac{nx^n}{1-x^n} \sim \frac{\pi^2}{6} \cdot \frac{1}{(1-x)^2} \quad \ldots (13{\cdot}23)$$

when $\to 1$.

It seems however, that possibly $\pi^3/12$ should be replaced by $\pi^3/6$ in formula (13.22), which is taken from "Encycl des sciences math." Tome I, vol. 3, p. 355. In fact, on p. 359, $n^r\zeta(r+1)$ is given as the "asymptotic mean-value" of the sum of the r^{th} powers of the divisors of a positive integer n, that is

$$\frac{1}{n}\sum_{h=1}^{n}\sum_{d:h}d^r \quad n^r\zeta(r+1),$$

a result attributed to E. Cesàro, L. Gegenbauer and L. Kronecker. Hence the result for $r=1$ should be $n\,\pi^2/6$, while (13.22) gives $n\,\pi^2/12$ ($n\frac{\pi^6}{6}$ is obviously a misprint. Another obvious misprint is "*quadratiques*" instead of "*cubiques*" on the same page (359) of the "Encycl." line 5 *f·b*,). I have neither seen any of the original memoirs in question nor as yet taken the trouble to verify the results, and I am therefore at present not in a position to judge, where the mistake lies.*

It is however easy to verify another formula, also given on p. 359 of the "Encycl.," *viz.*: The asymptotic mean value of the sum of the inverted values of the r^{th} powers of the divisors of n equals $\zeta(r+1)$.

In fact

$$\sum_{n=1}^{\infty}\frac{1}{n^r}\frac{x^n}{1-x^n}=\sum_{n=1}^{\infty}x^n\sum_{d:n}\frac{1}{d^r}, \qquad |x|<1.$$

Thus

$$(1-x)\sum_{n=1}^{\infty}\frac{1}{n^r}\frac{x^{n-1}}{1-x^n}=\frac{x.\frac{1}{1^r}+x^2.\left(\frac{1}{1^r}+\frac{1}{2^r}\right)+x^3\left(\frac{1}{1^r}+\frac{1}{3^r}\right)+\dots}{x+x^2+x^3+\dots},$$

and hence

$$-\frac{1}{n}\sum_{h=1}^{n}\sum_{d:h}\frac{1}{d^r}\sim\sum_{n=1}^{\infty}\frac{1}{n^{r+1}}=\zeta(r+1), \qquad \dots \quad (13{\cdot}24)$$

as $n\to\infty$ (since the left side is an increasing function of n, and cannot tend to infinity, thus having a definite limit).

If we put $\psi(n)=\phi(n)$ we get

$$\sum_{n=1}^{\infty}\frac{x^n}{1-x^n}\sum_{d:n}\mu(d)\phi\left(\frac{n}{d}\right)=\sum_{n=1}^{\infty}x^n\phi(n),\ |x|<1 \quad \dots \quad (13.3)$$

(*) Since the above was written, I have proved that Dirichlet's result is the correct one. The proof will be given in the following.

Now Dirichlet has proved, that

$$\sum_{h=1}^{n} \phi(h) = \frac{3n^2}{\pi^2} + O(n^\delta) \quad \ldots \quad \ldots \quad \ldots \quad (13.31)$$

where δ lies between 1 and 2, while according to F. Mertens

$$\sum_{h=1}^{n} \phi(h) = \frac{3n^2}{\pi^2} + O(n \log n). \quad \ldots \quad \ldots \quad (13.32)$$

Hence

$$(1-x) \sum_{n=1}^{\infty} \frac{x^{n-1}}{1+x+x^2+\ldots+x^{n-1}} \sum_{d\,:\,n} \mu(d) \phi\left(\frac{n}{d}\right)$$

$$= \frac{\phi(1)x + \phi(2)x^2 + \phi(3)x^3 + \ldots}{x + 2x^2 + 3x^3 + \ldots} \quad \ldots \quad (13.33)$$

whence

$$\sum_{n=1}^{\infty} \frac{x^n}{1-x^n} \sum_{d\,:\,n} \mu(d) \phi\left(\frac{n}{d}\right) \sim \frac{6}{\pi^2} \cdot \frac{1}{(1-x)^2} \quad \ldots \quad (13.34)$$

when $x \to 1$.

We know further that $p(n)$ is the numerical integral of $\mu^2(n)$.

Thus

$$\sum_{n=1}^{\infty} \frac{x^n}{1-x^n} \mu^2(n) = \sum_{n=1}^{\infty} p(n) x^n \quad , |x| < 1. \quad \ldots \quad (13.4)$$

Since now Mertens has proved that

$$\sum_{h=1}^{n} p(h) = \frac{6}{\pi^2} n \log n + O(n) \quad \ldots \quad \ldots (13.41)$$

or more accurately

$$\sum_{h=1}^{n} p(h) = \frac{6n}{\pi^2}\left(\log n + 2C - 1 + \frac{12}{\pi^2} \sum_{h=2}^{\infty} \frac{\log h}{h^2}\right)$$

$$+ O(\sqrt{n} \log n) \quad \ldots (13.411)$$

we have in the usual manner, and by the aid of (13.11) & 13·12),

$$\sum_{n=1}^{\infty} \frac{x^n}{1-x^n} \mu^2(n) \sim \frac{6}{\pi^2} \frac{1}{1-x} \log\left(\frac{1}{1-x}\right), \quad \ldots (13.42)$$

when $x \to 1$.

Combining (13.11) and (13·42), we have also

$$\sum_{n=1}^{\infty} \frac{x^n}{1-x^n}\left(\frac{\pi^2}{6}\mu^2(n)-1\right) = \sum_{n=1}^{\infty} x^n\left(\frac{\pi^2}{6}.\ p(n)-t(n)\right), \quad \ldots \quad (13.5)$$

from which we conclude, comparing (13.13) and (13.411), that

$$\sum_{n=1}^{\infty} \frac{x^n}{1-x^n}\left(\frac{\pi^2}{6}\mu^2(n)-1\right) \backsim \frac{12}{\pi^2}\sum_{h=2}^{\infty}\frac{\log h}{h^2}.\frac{1}{1-x} \quad \ldots \quad (13.51)$$

when $x \to 1$, or

$$\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{\pi^2}{6}\mu^2(n)-1\right) = \frac{12}{\pi^2}\sum_{h=1}^{\infty}\frac{\log h}{h^2}, \quad \ldots \quad (13.52)$$

provided the series to the left converges.

Putting $f(n) = (-1)^{\frac{1}{2}(n-1)}$, n odd
$f(n) = 0$, n even

we obtain

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n-1}}{1-x^{2n-1}} = \sum_{n=1}^{\infty}x^n\sum_{d':n}(-1)^{\frac{1}{2}(d'-1)}, \quad \ldots \quad (13.6)$$

d' taking the values of all *odd* factors of n.

Since we have

$$\left(\frac{x}{1-x}-\frac{x^3}{1-x^3}+\frac{x^5}{1-x^5}-\ldots\right) \sim \frac{\pi}{4}.\frac{1}{1-x}, \quad \ldots \quad \ldots \quad (13·61)$$

[Cesáro]
[Bromwich : p. 172, (21).]

as $x \to 1$, we obtain

$$\operatorname*{Lt}_{n\to\infty}\ \frac{1}{n}\sum_{h=1}^{n}\mathrm{E}(h) = \frac{\pi}{4}, \quad \ldots \quad \ldots \quad (13·62)$$

where

$$\mathrm{E}(n) = \sum_{d':n}(-1)^{\frac{1}{2}(d'-1)}, \quad \ldots \quad \ldots \quad (13·63)$$

(provided this limit exists).

Now, however, A. Berger * has proved, that

$$\operatorname*{Lt}_{m\to\infty}\ \frac{\mathrm{E}(1)+\mathrm{E}(3)+\ldots+\mathrm{E}(2m-1)}{2m-1} = \frac{\pi}{4}, \quad (13.631)$$

* Acta math, 9 (1886/7), p. 301.

from which we conclude the corresponding

$$\lim_{m\to\infty} \frac{E(2)+E(4)+\ldots+E(2m)}{2m}=0. \qquad \ldots \qquad \ldots \quad (13.64)$$

As a consequence of a formula (by J. Liouville†) we have also

$$\sum_{n=1}^{\infty} f(n)\frac{x^n}{1-x^n}=\sum_{n=1}^{\infty} x^n \sum_{d:n}\left(\frac{n}{d}\right)t(d)\ . \quad |x|<1 \qquad \ldots \quad (13{\cdot}7)$$

From the relation

$$\sum_{d:n} p(d)=t(n^2) \qquad \ldots \qquad \ldots \qquad \ldots \quad (13{\cdot}8)$$

it follows further for $f(n)=p(n)$, that

$$\sum_{n=1}^{\infty} \frac{p(n)x^n}{1-x^n}=\sum_{n=1}^{\infty} t(n^2)x^n,\ |x|<1 \qquad \ldots \quad (13{\cdot}81)$$

14. For the valnes $f(n)=\mu(n)$, $f(n)=\varphi(n)$, the right side in (12·1) becomes a rational function. We have in fact in the first case

$$\sum_{n=1}^{\infty} \frac{\mu(n)x^n}{1-x^n}=x,\ |x|<1, \qquad \ldots \qquad \ldots \quad (14{\cdot}1)$$

because of the well-known formula

$$\sum_{d:n} \mu(d)=0,\ n>1. \qquad (14{\cdot}11)$$

It is worth noticing that the same series is obtained by μ—inversion of the geometrical progression

$$\frac{x}{1-x}=\sum_{n=1}^{\infty} x^n.$$

If we sum the inverted series before rearranging the terms we obtain the formula (4 21). This is again a consequence of (14·11).

If we take the limit as $x\to 1$ after multiplication by $1-x$, the result is (11·13).

In the second case we have

$$\sum_{n=1}^{\infty} \frac{x^n}{1-x^n}\sum_{d:n}\mu(d)\left(\frac{n}{d}\right)=\sum_{n=1}^{\infty} nx^n,\ |x|<1. \qquad \ldots \quad (14{\cdot}2)$$

† J. math. pures appl.

But as $\sum_{d\,n}\phi(d)=n$, we have by (12·12)

$$\phi(n)=\sum_{d\,:\,n}'\mu(d)\left(\frac{n}{d}\right), \quad \ldots \quad \ldots \quad \ldots \quad (14{\cdot}21)$$

and hence

$$\sum_{n=1}^{\infty}{}'\frac{\phi(n)x^n}{(1-x)^n}=\frac{x}{(1-x)^2}. \quad \ldots \quad \ldots \quad (14{\cdot}22)$$

Hence we have also

$$\sum_{n=1}^{\infty}\frac{\phi(n)x^n}{1+x+\ldots+x^{n-1}}\sim\frac{1}{1-x} \quad \ldots \quad \ldots \quad (14{\cdot}23)$$

when $x\to 1$, which may be regarded as a complement to the equation

$$\frac{\zeta(s-1)}{\zeta(s)}=\sum_{n=1}^{\infty}{}'\frac{\phi(n)}{n^s},$$

in the theory of Dirichlet's series.

15. Let $$f(n)=\mu(n)\log n.$$

It appears, that the numerical integral in this case is

$$\sum_{d\,:\,n}'\mu(d)\log d=-\,\mathrm{V}(n), \quad \ldots \quad \ldots \quad (15{\cdot}1)$$

(a function which has been examined by N. V. Bugajev), where $\mathrm{V}(n)=\log p$, if n is a power of a prime p, $\mathrm{V}(n)=0$, when n contains two or more distinct prime factors, and $\mathrm{V}(1)=0$.

Consequently

$$\sum_{n=1}^{\infty}\mu(n)\log n\frac{x^n}{1-x^n}=-\sum_{n=1}^{\infty}\mathrm{V}(n)x^n,\ |x|<1, \quad \ldots \quad (15{\cdot}2)$$

from which follows

$$\sum_{n=1}^{\infty}\mu(n)\log n\frac{x^{n-1}}{1+x+\ldots+x^{n-1}}=-\frac{\mathrm{v}(1)x+\mathrm{v}(2)x^2+\mathrm{v}(3)x^3+\ldots}{x+x^2+x^3+\ldots} \quad (15{\cdot}21)$$

By Cesarò's theorem we see that the fraction on the right side cannot oscillate between wider limits, when $x \to 1$, than does $\frac{1}{n}\sum_{h=1}^{n} v(h)$ when $n \to \infty$. And as de la Vallee' Poussin*) and J Hadamard**) have proved, that the last fraction tends to 1, we have here a new verification of the theorem

$$\sum_{n=1}^{\infty} \frac{\mu(n) \log n}{n} = -1 \quad \ldots \quad \ldots \quad (15{\cdot}3)$***)$$

16. *The arithmetical function, whose numerical integral is* $\mu(n)$.

We have because of (12.14) for $\psi(n) = \mu(n)$

$$\sum_{n=1}^{\infty} \frac{x^n}{1-x^n} \sum_{d:n} \mu(d).\mu\left(\frac{n}{d}\right) = \sum_{n=1}^{\infty} x^n \mu(n), \; |x| < 1. \quad \ldots \quad (16.1)$$

Put for brevity.

$$\sum_{d:n} \mu(d)\mu\left(\frac{n}{d}\right) = \mu'(n). \quad \ldots \quad \ldots \quad (16.11)$$

By considering all different cases which may arise, we see that $\mu'(n)$ has the following properties:

$$\mu'(1) = 1.$$

$$\mu'(n) = 0,$$

if n is divisible by a cube other than unity.

$$\mu'(n^2) = 1, \text{ if } \mu(n) \neq 0,$$

that is if n is not divisible by a square other than unity.

$$\mu'(n) = (-2)^k, \text{ if } \mu(n) = (-1)^k,$$

that is if n is not divisible by a square other than unity, and k is the number of prime-factors in n.

$$\mu'(n_1 n_2^2) = (-2)^{k_1}$$

if $n_1 > 1, n_2 > 1$ are prime to one another, neither being divisible by a square other than unity, and k_1 is the number of prime-factors in n_1 ... ... (16·12)

Let now

$$\sigma(n) = \sum_{r=1}^{n} \mu(r) \quad \ldots \quad \ldots \quad \ldots \quad (16{\cdot}2)$$

---

* Ann. Soc. Scient. Bruxelles 20² (1895/6), p. 251.

**) Bull. Soc. math. France 24 (1896), p. 199.

***) C. R. Acad. Sc. Paris 129 (1899), p. 812.

It has been conjectured, that always

$$|\sigma(n)| < |\sqrt{n}|, \quad \ldots \quad \ldots \quad (16.21)$$

the proof of which would have far-reaching consequences, furnishing in particular a proof of the famous proposition of Riemann that the imaginary roots of the equation

$$\zeta(s)=0$$

all have $R(s)=\frac{1}{2}$, while it the latter conjecture were proved correct, it would follow, that for n large enough

$$|\sigma(n)| < n^{\theta},$$

where θ is a certain number less than $\frac{6}{7}$*). It would also follow, that the series in formula (11·12) is convergent for $R(s)>\theta$. *)

At present we know only, that the order of $\sigma(n)$ is not greater than that of $n\,e^{-a\sqrt{\log n}}$ [Landau *], which however is sufficient to find the value, if it exists, of the series

$$\sum_{n=1}^{\infty} \frac{\mu'(n)}{n}.$$

In fact we have by (16·1)

$$\sum_{n=1}^{\infty} \frac{x^{n-1}}{1+x+\ldots+x^{n-1}}\,\mu'(n)=\frac{\mu(1)x+\mu(2)x^2+\mu(3)x^3+\ldots}{x+x^2+x^3+\ldots}. \quad (16·3)$$

As $x \to 1$, the right side of this equation can only oscillate between limits, which are not greater in absolute value than those of

$$\frac{ne^{-a\sqrt{\log n}}}{n}$$

as $n \to \infty$, and as this last expression tends to zero, we may conjecture

$$\sum_{n=1}^{\infty} \frac{\mu'(n)}{n}=0. \quad \ldots \quad \ldots \quad (16·4)$$

From the general formula in the theory of Dirichlet's series

$$\sum_{n=1}^{\infty} \frac{h(n)}{n^s}=\sum_{n=1}^{\infty} \frac{f(n)}{n^s}\cdot\sum_{n=1}^{\infty} \frac{g(n)}{n^s}, \quad \ldots \quad (16·5)$$

*) E. Landau, Rend. Circ. Mat. Palermo 26 (1908), p. 264.

where the series are convergent, $f(n)$, $g(n)$ any given arithmetical functions, and

$$h(n)=\sum_{d:n} f(d)\, g\left(\frac{n}{d}\right), \qquad \ldots \qquad \ldots \quad (16{\cdot}51)$$

it follows for $f\equiv g\equiv\mu$

$$\sum_{n=1}^{\infty}{}' \frac{\mu'(n)}{n^s}=\left\{\sum_{n=1}^{\infty}{}' \frac{\mu(n)}{n^s}\right\}^2=\frac{1}{\zeta^2(s)} \quad \ldots \qquad \ldots \quad (16{\cdot}6)$$

Thus

$$\sum_{n=1}^{\infty}{}' \frac{\mu'(n)}{n^s}\cdot\sum_{n=1}^{\infty}{}' \frac{t(n)}{n^s}=1, \qquad \ldots \qquad \ldots \quad (16{\cdot}61)$$

[Bromwich, p. 494 Ex. 45 (2)].

and consequently, if we put

$$\rho(n)=\sum_{n:d}{}' \mu'(d)\, t\left(\frac{n}{d}\right)=\sum_{d:n}{}' \mu'\left(\frac{n}{d}\right) t(d), \qquad \ldots \quad (16{\cdot}62)$$

we have by (16·51) $$\sum_{n=1}^{\infty}{}' \frac{\rho(n)}{n^s}=1, \quad \ldots \qquad \ldots \qquad \ldots \quad (16{\cdot}7)$$

from which we conclude $\rho(n)\equiv 0$, for $n>1$ $\qquad \ldots \qquad \ldots \quad (16{\cdot}1)$

The formula (16·1) could also have been obtained through μ—inversion of (14·1).

(To be continued.)

Double Lines.

By M. Bhimasena Rao.

[This note is in continuation of the Author's Note on "Double Points" on page 19, Vol. IV of the Journal.]

Let ABC and A′B′C′ be any two triangles and P and P′ the mean centres of A, B, C and A′, B′, C′ respectively for any common system of multiples. The line joining any two points P and Q corresponds to the join of P′ and Q′. The self corresponding lines may be called the double lines of the triangles. The line at infinity is evidently a double line.

2. *To find the double lines.*

Let D be the double point of the triangles. Divide AB and A′ B′ in the same ratio at X and X′. DX and DX′ are corresponding lines. X and X′ divide AB and A′ B′ homographically. Hence D X and D X′ are conjugate rays of a homographic pencil whose double lines are the double lines of the triangles.

3. The double lines are evidently the tangents from D to the parabola inscribed in the quadrilateral ABB′A′. Hence the three parabolas inscribed in the quadrilaterals ABB′A′, BCC′B′ and CAA′C′ have two tangents in common which are the double lines of ABC and A′B′C′, and the line at infinity which is also a common tangent of the parabolas is also a double line. The intersection of two double lines is a double point. It follows therefore that any two triangles have *in general** three double lines, and three double points two of which are at infinity. Denote these two points by D_1 and D_2.

4. When ABC and A′ B′ C′ are directly similar triangles, D_1 and D_2 are the circular points. The following properties can be easily inferred by orthogonal projection.

(i) The conics ABC D_1 D_2 and A′ B′ C′ D_1 D_2 are corresponding curves. If X be any point on the former, lines through A′, B′, C′ parallel to AX, BX, CX intersect at a point X′ on the latter. X and X′ are not corresponding points of the triangles.

(ii) If P, P′ and Q, Q′ be pairs of corresponding points, and P P′ and Q Q′ intersect in O, the conics PQD D_1 D_2 and P′ Q′ D D_1 D_2 are corresponding curves passing through O and are the loci of corresponding points whose join passes through O.

5. *To find the condition that the double lines are at right angles.*

In this case D is on the directrices of the three parabolas AB B′A′, BC C′B′, CA A′B′, *i.e.*, on the radical axes of the pairs of circles described on AB′, A′B; BC′, B′C; CA′, C′A; as diameters. If X and X′ be the mid-points of AB′ and A′B, we have

$$DX^2 - DX'^2 = \frac{1}{4}(AB'^2 - A'B^2),$$

$$DA^2 + DB'^2 = 2\,DX^2 + \frac{AB'^2}{2},$$

$$DA'^2 + DB^2 = 2\,DX'^2 + \frac{A'B^2}{2}$$

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* In particular, when A A′, B B′ and C C′ are parallel, any line parallel to these is a double line and any point on the axis of perspective of ABC and A′ B′ C′ is a double point.

$$\therefore \quad (DA^2-DB^2)-(DA'^2-DB'^2)=2\,(DX^2-DX'^2)+\tfrac{1}{2}(AB'^2-A'B^2)$$
$$=AB'^2-A'B^2.$$

Adding two similar results, we have

$$AB'^2+BC'^2+CA'^2=A'B^2+B'C^2+C'A^2,$$

showing that the triangles ABC and A′B′C′ are orthologic.

Given that ABC and A′B′C′ are orthologic triangles, it can be shown by a rearrangement of the steps that the radical axes of the three pairs of circles described on AB′ and A′B ; etc. as diameters co-intersect, but it does not follow that the point of intersection is the double point of the triangles.

That the triangles should be orthologic when the double lines are at right angles follows immediately by the application of the first property of § 4. $ABCD_1D_2$ is now a rectangular hyperbola which passes through H the orthocentre of ABC. Therefore lines through A′, B′, C′ parallel to AH, BH, CH, *i.e.*, perpendicular to BC, CA, AB are concurrent on the rectangular hyperbola $A'B'C'D_1D_2$ and for a similar reason perpendiculars from A, B, C, on B′C′, C′A′, A′B′, are concurrent on $ABCD_1D_2$.

6. Given that ABC and A′B′C′ are orthologic triangles, to show the double lines are right angles.

Let the perpendiculars from A, B, C on the corresponding sides of A′B′C′ intersect at S and the perpendiculars from A′B′C′ on the sides of ABC, at S′. Then S and S′ are corresponding point of the triangles.* Let L and L′ be points at infinity on BC and B′C′ and AS, A′S′ intersect the line at infinity in M and M′. Since corresponding lines cut the line at infinity homographically and D_1 and D_2 are the double points of this range we have

$$(D_1D_2,\ LL')=(D_1D_2,\ MM').$$
$$\therefore \quad (D_1D_2,\ LL')=(D_2D_1,\ M'M).$$
$$\therefore \quad (D_1D_2,\ LM',\ L'M) \text{ is in involution.}$$

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* This is pointed out by Gallatly in his *Modern Geometry*, *vide* page 56. The following is a geometrical proof :—

Let T be the isogonal conjugate of S and XYZ the pedal triangle of T with respect to ABC. S and T are corresponding points of ABC and XYZ. (Apply Ex. 7 (*a*) and 11 pp. 88 and 89 of M'clelland's Geometry of the circle). The sides of XYZ being at right angles to SA, SB, SC, the triangles A′B′C′ and XYZ ard homothetic, TX, TY, TZ being parallel to S′A′, S′B′, S′C′, T and S′ are correspone-ing points of XYZ and A′B′C′. Therefore S and S′ are corresponding points of ABC and A′B′C′.

But L and M′ are points at infinity in direction at right angles as also L′ and M. Therefore D_1 and D_2 are points at infinity in direction at right angles, *i.e*, the double lines are at right angles.

Since ABC SD_1D_2 is a rectangular hyperbola, the double lines are parallel to the pedal lines of the extremeties of the circum-diameter of ABC through the isogonal conjugate of S.

7. *Double point of perspective triangles.*

Consider a triangle ABC and its conjugate triangle A′B′C′ with respect to a circle whose centre is D. It is well known that the triangles are in perspective and also orthologic, the points S and S′ determined as in § 6 being coincident at the centre D which shows that the centre is the double point, and the rectangular hyperbolas ABCD and A′B′C′D intersect at the centre of perspective and also on the line at infinity. By orthogonal projection we have.

(1) The double point of two triangles in perspective is the centre of the conic with respect to which they are conjugate (Q. 367).

(2) The double lines are conjugate diameters of the above conic (Q. 374, (2)), and if O is the centre of perspective.

(3) The conics ABCDO and A′B′C′DO intersect the line at infinity in the remaining doule points (Q. 374, (1)) and are the loci of corresponding points whose join passes through O, by the second property of § 4, (Q. 374, (4)). If, in addition, ABC and A′B′C′ be orthologic, S and S′ being the points of concurrence of the perpendiculars from the verticies of either on the corresponding sides of the other, the conics ABCDO and A′B′C′DO are rectangular hyperbolas passing respectively through S and S′. Since S and S′ are corresponding points, they are collinear with the centre of perspective (Q. 643). The first result of Q. 340 is an application of this property.

8. Besides the application and illustrations of the theory of double points and lines given in the above questions set in the Journal, the following are added.

(1) The double lines of the pedal triangles of any two points P and P′ with respect to ABC are the pedal lines of the points in which PP′ intersects the circumcircle of ABC. When PP′ passes through the circumcentre, the pedal lines are at right angles, and therefore the pedal triangles of P and P′ are orthologic; and conversely if the pedal triangles of P and P′ are orthologic, PP′ passes through the circumcentre.

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(2) If XYZ and X′Y′Z′ are the pedal triangles of P and P′ on a circum-diameter of a triangle the radical axes of the three pairs of circles described on XY′ and X′Y, etc., as diameters are concurrent on the nine point circle at the orthopole of PP′ (Q. 239 is a particular case of this property).

(3) Let P, P′ be corresponding points and L, L′ corresponding lines of two orthologic triangles F, F′. Then the perpendicular from P on L′ considered as a line of F corresponds to the perpendicular from P′ on L. Inversely similar triangles are orthologic. For inversely similar triangles we have an additional property, viz., the parallel through P to L′ corresponds to the parallel through P′ to L, given by Lachlan: *Modern Geometry*, Ex. 1, § 222.

(4) Let A′B′C′ be the pedal triangle of Pwith respect to ABC. To the circumcircle of ABC corresponds a circum-ellipse of A′B′C′. This ellipse is the locus of orthopoles of lines through P with respect to ABC. The double lines being at right angles give the directions of the axes of the ellipse.

(5) If two conics having the same axes are such that triangles inscribed in one are circumscribed to the other, the normals at the points of contact are concurrent. [A well known property.]

For if ABC be an inscribed triangle and A′, B′, C′ the points of contact, ABC and A′B′C′ are conjugate triangles with respect to the second conic, the double point is the common centre of the conics which therefore are corresponding conics, the common axes being conjugate diameters of the conics are the double lines, and these being at right angles, the triangles ABC and A′B′C′ are orthologic, etc. If a concentric conic be inscribed in A′B′C′, the normals at the points of contact are also concurrent.

(6) To project orthogonally orthologic triangles into orthologic triangles, we have only to take the base line parallel to a double line. If inversely similar triangles be projected into orthologic triangles, these latter will also be inversely similar, for the double points and any two corresponding points on the line at infinity, of inversely similar triangles form a harmonic range—a property unaltered by projection.

(To be concluded).

SHORT NOTES.

Note on Question 735.

1. The following *general result* may be of interest :

Let (r_n, θ_n) be the point of the nth pedal corresponding to a point (r, θ) of a curve. Then

$$\theta = \theta_n + n(\tfrac{1}{2}\pi - \phi) \; ; \; r_n = r \sin^n\phi.$$

Now, if C_n is the entire circumference of the nth pedal, we have

$$C_n = \int r_{n+1} d\theta_{n+1} = \int r \sin^{n+1}\phi (d\theta + \overline{n+1} d\phi).$$

[J.I.M.S.. 1916, p. 137.)

Also
$$C_{n+2} = \int d s_{n+2} = \int r_{n+2} \operatorname{cosec} \phi d\theta_{n+2}$$
$$= \int r \sin^{n+1}\phi (d\theta + \overline{n+2} d\phi)$$

$\therefore$
$$C_{n+2} - C = \int r \sin^{n+1}\phi \, d\phi,$$

and
$$(n+2)\, C_n - (n+1) C_{n+2} = \int r \sin^{n+1}\phi \, d\theta \; ;$$

which may also be written

$$\left.\begin{aligned} (n+1)C_n - nC_{n-2} &= \int r \sin^{n+1}\phi d\psi, \\ (n+2)C_n - (n+1)C_{n+2} &= \int \rho \sin^{n+2}\phi d\psi \end{aligned}\right\}$$

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2. For the particular case of Q. 735, we write

$$d\psi = \frac{abd\alpha}{r'^2},$$

where α is the eccentric angle of the point (r, θ) and r' is the radius conjugate to r. Thus

$$(n+1)C_n - nC_{n+2} = \int \frac{r \cdot p^{n+1}}{r^{n+1}} \cdot \frac{abd\alpha}{r'^2} \cdot$$
$$= \int \frac{(ab)^{n+2} d\alpha}{r^n r'^{n+2}};$$

and
$$(n+2)\, C_n - (n+1) C_{n+2} = \int \frac{r'^n}{ab} \cdot \frac{p^{n+2}}{r^{n+2}} \cdot \frac{abd\alpha}{r'^2} \cdot$$
$$= \int \frac{(ab)^{n+2} d\alpha}{r^{n+2} \cdot r'^{n+1}} \cdot$$

Putting $n=0,-1,-2$ in the above, we have

$$C_0=\int\frac{a^2b^2da}{r'^3}$$

$$C_1=\int\frac{dbrda}{r'^2}$$

$$2C_0-C_{-2}=\int\frac{r^2}{r'}da$$

$$2C_0-C_2=\int\frac{a^2b^2}{r^2r'}da$$

$$C_{-1}=\int\frac{abda}{r}$$

$$C_0=\int r'da$$

whence $$C_1+C_{-1}=ab(a^2+b^2)\int\frac{da}{rr'^2}$$

$$3C_0-C_{-2}=(a^2+b^2)\int\frac{da}{r'}$$

$\therefore$ $$C_{-1}(C_1+C_{-1})=(3C_0-C_{-2})(2C_0-C_2)$$

since $$\int\frac{da}{r}\int\frac{da}{rr'^2}=\int\frac{da^1}{r'}\int\frac{da^1}{r^2r'}=\int\frac{da}{r'}\int\frac{da}{r^2r'}.$$

M. T. NARANIENGAR.

A Note on Question 837.

1. In the Annual Progress Report for the year 1915—16 of the Superintendent, Hindu and Buddhist monuments, Northern Circle, mention is made of a magic square found on the underside of a fallen lintel at the shrine, locally known as *chota Surang*, at Dudhai, Jhansi District. This temple probably belongs to the first half of the eleventh century A. D. The Square is as follows :—

7	12	1	14
2	13	8	11
16	3	10	5
9	6	15	4

(1)

The Superintendent remarks :—

"Mathematically it is interesting as possessing the following properties:— (i) the sum of each row, each column and each diagonal is 34, (ii) the sum of all the numbers in each sub-square also 34."

Since (i) is true not only of the leading diagonals, but also of broken diagonals, the square is of the type known as pandiagonal (see Ball's Mathematical Recreations,* p. 156). (ii.) however is not an additional property as is evidently supposed by the writer, but is a consequence of (i). In fact if we divide any even Magic square into four groups of cells which we may represent by A,B,C,D as below

A	B
C	D

and if k be the constant sum of rows and columns, we clearly have

$$A+B=\frac{n.k}{2}=A+C \quad \therefore B=C \text{ and } A=D.$$

but in general $A \neq C$ or B

If however the square is of the pandiagonal type, it is easy to show that we have

$$A=B=C=D=\frac{nk}{4}$$

Now in a pandiagonal square we may clearly alter the order of either rows or columns cyclically without losing the properties of the square. Hence it follows that $A=nk/4$ where A is any subsquare of $nk/4$ cells (n^2 = total no. of cells), in this case *e.g.* the four centre cells. It has seemed worth mentioning this property of pandiagonal squares, since it is not mentioned by Ball and is clearly unknown to the writer of the Report, as also to the writer of the Scientific notes in Chambers' Journal where I first came across the square.

2. Squares of the type of (I) are directly connected with those of the type required as a solution to Mr. Malhari Rao's question No. 837, and as a matter of fact the above square leads immediately to a solution of Q. 837 quite distinct from the set of solutions given in J. I. M. S. vol. ix. p. 16!.

Making one alteration in rows, one in columns and then writing rows as columns, (I) becomes.

1	15	10	8
12	6	3	13
7	9	16	2
14	4	5	11

(II)

* The references are all to the fifth Edition.

Subtracting unity from each element of (II) and then replacing each element n(say) so obtained by 2^n we obtain the following square which is clearly a solution of Q. 837.

1	2^{14}	2^9	2^7
2^{11}	2^5	2^2	2^{12}
2^6	2^8	2^{15}	2
2^{13}	2^3	2^4	2^{10}

(III)

This solution differs from any of the doubly infinite set obtained in J. I. M. S. vol. ix p. 161 in that in (III) all the 16 elements are different, whereas in the previous solution there were only 8 distinct elements each repeated. Denoting the elements by $a_1, a_2, \ldots b_1, b_2, \ldots$ and $a_1.a_2.a_3.a_4 = k$ we have 16 relations connecting 17 quantities but the equations are in reality equivalent to only 14 independent relations; if therefore, we assign particular values to any two of the quantities we can still assign any arbitrary value to the quantity k and obtain a solution. It does not follow however that the solution so obtained will be in positive integers. For example suppose k is prime, then the only possible solution will have one element in each row $= k$ and all the others unity; it must be one of the solutions of the problem of placing 4 queens on a chessboard of 16 cells so that no queen can capture any other—there are two solutions of this problem (see Ball, loc. cit. *p.* 114), but although they make the product of rows and columns $= k$, they break down for the diagonals, hence there is no solution in positive integers if k is prime.

The solution previously given in J. I. M. S. is not the most general solution, the values obtained $\lambda_1 + \lambda_2 = 2$ are only one set of possible values.

3. The question arises whether the solution (III) is the only solution in which the cells are occupied by distinct integers. It is not so as may be seen as follows:—

The square (II) is the sum of the two squares

0	$3m$	2	m
$2m$	m	0	$3m$
m	$2m$	$3m$	0
$3m$	0	m	$2m$

and

1	3	2	4
4	2	3	1
3	1	4	2
2	4	1	3

provided $m=4$; but each of these squares is a pandiagonal magic square for all values of m, and hence so is their sum. But in the sum the cells a_1 and c_4 are occupied by the nos. 1 and 2 for all values of m, hence adding the two squares, subtracting unity from each element and replacing each element so obtained by the corresponding power of 2 we obtain the square

1	2^{8m+2}	2^{2m+1}	2^{m+3}
2^{3m+3}	2^{m+1}	2^{2}	2^{3m}
2^{m+2}	2^{2m}	2^{3m+2}	2
2^{3m+1}	2^{3}	2^{m}	2^{2m+2}

which satisfies the conditions of the problem whatever the value of m and provided $m>3$ the cells are occupied by 16 distinct integers; in this way we get a singly infinite system of such solutions.

4. Any factor of k must occur as a factor at least once in every line, column and diagonal, but it has been shown above that if it occurs once only no solution is possible; it must therefore occur at least twice and hence for a solution in positive integers k must be a perfect square. But since $c_4=2$, 2 is a factor of k and there 4 is a factor of k, hence it is a necessary condition that k is of the form $4p^2$ (This appears to be proved by the solution previously published where it is found that $k=4b_4^2.d_4^2$, but since this solution was not perfectly general it seemed desirable to give a general proof.)

R. J. Pocock.

Astronomical Notes.

The nearest Star.

For three quarters of a century *alpha centauri* has been the nearest known star. Mr. Innes at Johannesburg has now discovered a star of the eleventh magnitude, the distance of which is almost exactly the same as that of *alpha centauri*. As its proper motion is of approximately the same amount and in the same direction, it seems possible that the two stars are physically connected although the distance separating them is very great, *viz.* 20° 12′.

The particulars of the two stars are given below :—

	Alpha centauri.	New Star.
Parallax ...	0·″756	0·″755
Proper Motion (in arc)	3·″68	3.″76
Direction ...	2814°	2827°

This star is intrinsically the faintest star known.

About a year ago Prof. Barnard discovered a star with a larger proper motion than any previously known. These discoveries show that we still have much to learn about even the nearer stars.

The Planets.

Most of the planets are favourably situated for observation at this time of year.

Below are given the positions on March 15th and the constellations in which the planets will be found.

Planet.	R. A.	Decln.	Constellation.
Mercury ...	$23^h\ 49^m$	... 2° 35′ S	... Pisces.
Venus ...	21 8	... 11 9 S	... Aquarius.
Mars ...	11 42	... 6 1 N	... Virgo.
Jupiter ...	4 13	... 20 40 N	... Taurus.
Saturn ...	8 43	... 19 7 N	... Cancer.
Uranus ...	21 52	... 13 43 S	... Capricornus.
Neptune ...	8 27	... 18 59 N	... Cancer.

The sun's position on the same date is R. A. $23^h\ 38^m$ Decl. 2° 20′ S. in the constellation Aquarius. It will be seen that Venus and Uranus are morning stars while all the others are evening stars, though at this date Mercury will be too close to the sun to be visible and Mars will just rise at sunset.

R. J. Pocock.

SOLUTIONS

Question 757.

(S. Malhari Rao) :—Give a solution in positive integers of the equations.

(1) $x^4-6y^2=1$, (2) $x^{12}-17y^2=x^6$.

Remarks by S. V. Venkatachala Iyer.

(1) $x^4-6y^2=1$, Let $x^2=z$,

Then $z^2-6y^2=1$, To solve this in positive integers we express $\sqrt{6}$ as a continued fraction ; we have.

$$\sqrt{6}=2+\frac{1}{2}+\frac{1}{4}+\frac{1}{2}+\frac{1}{4}+\ldots\ldots$$

The successive convergents are $2, \frac{5}{2}, \frac{22}{9},\ldots$—

$\therefore$ $z=5, y=2$ is obviously a solution of x^2-6y^2, but as this does not give the value of z a perfect square, as it should be, we have to select such a value for z, as will be a perfect square, from the general solution,

$$z=\tfrac{1}{2}\{(5+2\sqrt{6})^n+(5-2\sqrt{6})^n\}.$$

Putting $n=2$, we get $z=49$, a perfect square.

$\therefore$ The value of $x=7$, and the corresponding value of y is 20.

(2) $x^{12}-17y^2=x^6$. Dividing out by x^6, we have

$$x^6-17\frac{y^2}{x^6}=1 \quad \text{Put } x^3=\xi,\ y=\xi\eta$$

Then $\xi^2-17\eta^2=1$. To get values for ξ, η, in positive integers, we see $\sqrt{17}=4+\frac{1}{8+8+}\cdots$, and the successive convergents are $4, \frac{33}{8}, \frac{268}{65},\ldots$ Of these $\xi=33, \eta=8$, satisfies $\xi^2-17\eta^2=1$. But since 33 is not a perfect cube, we have to consider the general solution given by

$$\xi=\tfrac{1}{2}\{(33+8\sqrt{17})^n+(33-8\sqrt{17})^n\}$$

$$\eta=\frac{1}{2\sqrt{17}}\left\{(33+8\sqrt{17})^n-(33-8\sqrt{17})^n\right\}$$

and get a value for ξ which is a perfect cube, by properly chosing n and the corresponding value of η also can be found ; The relations $x^3=\xi$, $y=\xi\eta$, give x and y.

Question 761.

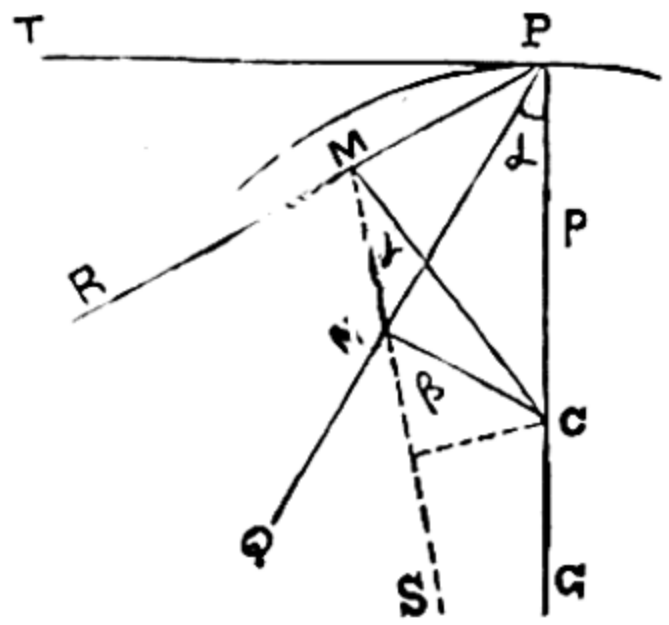
(MARTYN M. THOMAS) :—Apply the method of Instantaneous centre to solve the following—

"At every point of a plane curve a line is drawn making a given angle with the normal, and let the envelope of these lines be termed the α evolute. Prove that the β. evolute of the α evolute of any curve is the α evolute of the β evolute of the curve"

Solution by S. R. Ranganathan and the Proposer.

Let C be the centre of curvature at P to the given curve. Let $CPQ=\alpha$ and $CPR=\beta$. Draw PT the tangent atP to the given curve.

Consider PT, PR, PQ and PG as four rods rigidly connected, and let PT move tangentially to the given curve.



Then, as P moves on the curve to its consecutive position, PT, PR, PQ, PG will have moved to their consecutive positions, so that

PT	moves	tangentially	to the given-curve,
PR	„	„	to β-evolute,
PQ	„	„	to α-evolute,
PG	„	„	to evolute proper,

Thus the centre of curvature C is the instantaneous centre, and thus it becomes the centre of curvature for the α-evolute and β-evolute as well.

Hence the perpendiculars from C on PQ and PR become the ρ's for these α and β-evolutes, and M, N the feet of these perpendiculars are, consequently, the points of contact.

Join MN and produce it to S. Then MNCP being a cyclic quadrilateral, $NMC=\alpha$, and $SNC=\beta$.

$\therefore$ The curve enveloped by MN becomes the α-evolute of the β-evolute, and is also the β-evolute of the α-evolute.

The foot of the $\perp$ from C on MN is the point of contact, and corresponds to P.

Thus as P moves along the given curve, PQ envelopes the α-evolute PR envelopes the β-evolute and MN the α-evolute of the β-evolute and vice versa.

Question 815.

(Martyn M. Thomas, M.A.):—If

$$\int_\theta \frac{dx}{ax^2+bx+c} = \sum \frac{x^n}{n} k_n$$

prove that $$2a\,k_{n-1}+b\,k_n+n\,k_{n+1}=0.$$

Solution by K. B. Madhava, R. J. Pocock, H. R. Kapadia, A. A. Krishnaswami Aiyangar, S. R. Ranganathan, C. Krishnamachary, L. S. Vaidyanathan, V. M. Gaitonde, K. A. Erady, S. V. Venkatachalayya, L. N. Bhatt, B.A., K. Santhanam, K. K. R. Aiyar, C. Bhaskaraiya, K. R. Rama Iyer, and S. Gangadharan.

Let $$y=\int exp\,(-ax^2-bx-c)\,dx=\sum \frac{x^n}{n} k_n,$$

then $$y'=exp\ (-ax^2-bx-c)=\Sigma\, x^{n-1} k_n,$$

and $$y''+(2ax+b)\,y'=\Sigma\,(n-1)\,k_n\,x^{n-2}.$$

Substituting

$$\Sigma\,(n-1)\,k_n\,x^{n-2}+(2ax+b)\,\Sigma\,x^{n-1}\,k_n=0$$

and collecting the co-efficient of x^{n-1} we have

$$n\,k_{n+1}+b\,k_n+2\,a\,k_{n-1}=0.$$

Question 816.

(Martyn M. Thomas, M.A.):—If k be the curvature of an ellipse and

$$a_0=\frac{1}{2!}k,\ a_1=\frac{1}{3!}\frac{dk}{ds},\ a_2=\frac{1}{4!}\frac{d^2k}{ds^2},$$

show that

$a_0^{-\frac{8}{5}}\,(4a_0^4-5a_1^2+4a_0\,a_2)$ has the constant value $\left(\frac{2}{\alpha\beta}\right)^{\frac{1}{5}}$ at all points of the ellipse, α and β being the semi-axes.

Solution (1) *by K. B. Madhava*; (2) *by K. A. Erady, M.A.*,
(3) *K. K. R. Aiyar and several others.*

(1) We know $\frac{d^2}{dx^2}\left(\frac{d^2y}{dx^2}\right)^{-\frac{2}{3}} = \frac{2C}{\Delta^{\frac{1}{3}}}$ for the general conic

$$= \frac{-2}{(\alpha\beta)^{\frac{2}{3}}} \text{ from invariants.}$$

Now $$\frac{dy}{dx} = \tan\psi.$$

$$\therefore \left(\frac{d^2y}{dx^2}\right)^{-\frac{2}{3}} = k^{-\frac{2}{3}}\cos^2\psi,$$

$$\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)^{-\frac{2}{3}} = -\frac{2}{3}\,k^{-\frac{5}{3}}\frac{dk}{ds}\cos\psi - 2k^{\frac{1}{3}}\sin\psi$$

and
$$\frac{d^2}{dx^2}\left(\frac{d^2y}{dx^2}\right)^{-\frac{2}{3}} = -\frac{2}{9}\,k^{-\frac{8}{3}}\left(-5\left(\frac{dk}{ds}\right)^2 - 2 + 3k\frac{d^2k}{ds^2} + 9k^4\right)$$

$$= -2^{\frac{1}{3}}\,(a_0)^{-\frac{8}{3}}\,(-5a_1^2 + 4a_0a_2 + 4a_0^4).$$

Hence

$$a_0^{-\frac{8}{3}}\,(4a_0^4 - 5a_1^2 + 4a_0a_2) = \text{constant and equal to } \left(\frac{2}{\alpha\beta}\right)^{\frac{1}{3}}.$$

(2) Let tbe equation to the ellipse referred to the tangent and normal at a point be $2y = ax^2 + 2hxy + by^2$.

Then
$$y_1 = (ax + hy) + (hx + by)y_1$$
$$y_2 = (a + hy_1) + (hx + by)y_2 + (h + by_1)y_1,$$
$$y_3 = hy_2 + (hx + by)y_3 + 2(h + by_1)y_2 + by_1y_2$$
$$y_4 = hy_3 + (hx + by)y_4 + 3(h + by_1)y_3 + 3by_2^2 + by_1y_3.$$

At the origin $y_1 = 0$, $y_2 = a$, $y_3 = 3ah$, $y_4 = 3a^2b + 12ah^2$.

Also
$$\frac{dx}{ds} = (1 + y_1^2)^{-\frac{1}{2}}$$

Hence
$$k = \frac{1}{\rho} = y_2(1 + y_1^2)^{-\frac{3}{2}}$$

$$\frac{dk}{ds} = y_3(1 + y_1^2)^{-2} - 3(1 + y_1^2)^{-3}y_1y_2$$

$$\frac{d^2k}{ds^2} = y_4(1 + y_1^2)^{-\frac{5}{2}} - 3(1 + y_1^2)^{-\frac{7}{2}}\,y_2^2 + y_1(\quad)$$

$\therefore$ At the origin $k=a$, $\frac{dk}{ds}=3ah$ and $\frac{d^2k}{ds^2}=3a^2b+12ah^2-3a^3$.

Hence in the notation of the question

$$a_0=\frac{a}{2},\ a_1=\frac{ah}{2},\ a_2=\frac{a^2b+4ah^2-a^3}{8}$$

$\therefore$ $$4a_0^4-5a_1^2+4a_0a_2=\frac{a^2(ab-h^2)}{4}.$$

But if α, β are the semi-axes of $ax^2+2hxy+by^2=2y$,

$$\frac{1}{\alpha^2\beta^2}=\frac{(ab-h^2)^2}{a^2}.$$

Hence $a_0^{-\frac{8}{3}}(4a_0^4-5a_1^2+4a_0a_2)=\left(\frac{a}{2}\right)^{-\frac{8}{3}}(ab-h^2)$

$$=\left(\frac{2}{\alpha\beta}\right)^{-\frac{8}{3}}.$$

(3) We have $2a_0=k=\frac{d\psi}{ds}=\frac{1}{\rho}=\frac{p^2}{\alpha^2\beta^2}$ ... ... (i)

$$\frac{1}{k}\frac{dk}{ds}=\frac{3}{p}\cdot\frac{dp}{d\psi}\cdot k$$

so that $$\frac{dp}{d\psi}=\frac{p}{2}\left(\frac{a_1}{a_0^2}\right) \quad \ldots \quad \ldots \quad \ldots \text{ (ii)}$$

Also $\left(\frac{d^2k}{ds^2}\div\frac{dk}{ds}\right)=\frac{4a_2}{a_1}=\frac{d}{ds}\log\left[\frac{3k^2}{\rho}\frac{dp}{d\psi}\right]$

$$=\left\{\left(\frac{d^2p}{d\psi^2}\Big/\frac{dp}{d\psi}\right)+\frac{6}{p}\frac{dp}{d\psi}-\frac{1}{p}\frac{dp}{d\psi}\right\}k$$

$$=\left[\frac{\rho-p}{\frac{p}{2}\frac{a_1}{a_0^2}}+\frac{5}{2}\frac{a_1}{a_0^2}\right]2a_0, \text{ from (ii) since } \rho=p+\frac{d^2p}{d\psi^2}.$$

That is $$\frac{4a_2}{a_1}\cdot\frac{a_1}{a_0^2}=4a_0\left(\frac{\rho-p}{p}\right)+\frac{5a_1^2}{a_0^4},$$

or $$[4a_0a_2-5a_1^2+4a_0^4]=4a_0^4\cdot\frac{1}{2a_0}\cdot(2\alpha^2\beta^2a_0)^{-\frac{1}{3}}$$

$$=2^{\frac{2}{3}}a_0^{\frac{8}{3}}(\alpha\beta)^{-\frac{2}{3}}.$$

Hence $$a_0^{-\frac{8}{3}}[4a_0a_2-5a_1^2+4a_0^4]=\left(\frac{2}{\alpha\beta}\right)^{\frac{1}{3}}$$

is the differential equation to a conic in the intrinsic form.

———

Question 817.

(S. Malhari Rao) :—If $x\mathrm{C}_3$ is equal to the continued product of three primes, whose sum is 327, find x.

Solution by Martyn M. Thomas, M.A., and K. B. Madhava.

$$x\mathrm{C}_3=\frac{x(x-1)(x-2)}{1\times 2\times 3}$$

∴ We have the following cases :

$$x+\frac{x-1}{2}+\frac{x-2}{3}=327\ ;\ x+\frac{x-1}{3}+\frac{x-2}{2}=327$$

$$\frac{x}{2}+\frac{x-1}{3}+(x-2)=327\ ;\ \frac{x}{2}+(x-1)+\frac{x-2}{3}=327$$

$$\frac{x}{3}+\frac{x-1}{2}+(x-2)=327\ ;\ \frac{x}{3}+(x-1)+\frac{x-2}{2}=327.$$

From the first case, we have $x=179$; all the others do not give integral values of x.

∴ The primes, as obtained from the first, are 179, 89 and 59.

Question 818.

(S. Malhari Rao) :—Complete the following magic squares by inserting prime numbers in vacant cells :—

(*a*)

	1	
13		
		7

(*b*)

			47
		19	
	61		
53	83	1	43

(*c*)

		1		
		3		
103	97	449	43	37
		137		
		139		

Solution by R. J. Pocock.

I assume that the proposer requires the cells to be occupied by distinct integers.

(*a*)

Consider the square

$\alpha+6$	$1+0$	$\beta+12$
$1+12$	$\beta+6$	$\alpha+0$
$\beta+0$	$\alpha+12$	$1+6$

It will clearly be magic provided $\alpha+\beta+1+18=3\beta+18$ *i.e.* $\alpha+1=2\beta$; we have then to find two sets of 3 primes each in A. P. (common difference=6) such that the first of one set *plus* unity is double the first of the other set. If the numbers are to be distinct we easily find that $p=31$ satisfies all the conditions.

(*b*)

Consider the sqnare

a_1	a_2	a_3	47
b_1	b_2	19	b_4
c_1	6_1	c_3	c_4
53	83	1	43

we have $b_2+c_3=a_2+a_3+4$ assume $b_2=a_2+2, c_3=a_3+2$ then we have $a_2=17$, $b_2=19$, $a_3=79$, $c_3=81$.

$$\therefore \quad a_1=180-43-b_2-c_3=37.$$

$$\therefore \qquad b_1+c_1=90=b_4+c_4$$

and $$b_1+b_4=142\,;\ c_1+c_4=38$$

The only solution of $c_1+c_4=31$ in primes, other than those already assigned to cells is 7, 38 therefore b_1 and b_4 must be 83, 59.

Hence the sqnare is completely determined.

(*c*) We notice that the pairs of cells already filled which are symmetrically situated *w-r-t* the centre add up to 140. This suggests filling the remaining cells in the leading diagonals in a similar manner. One arrangement is shown below

79	a_2	1	a_4	7
b_1	67	3	53	$b5$
103	97	449	43	37
d_1	87	137	73	$5d$
93	e_2	139	e_4	61

we then have
$$\left.\begin{aligned} a_2+a_4&=602\\ e_2+e_4&=436\\ a_2+e_2&=478\\ a_4+e_4&=560 \end{aligned}\right\} \text{ and } \left\{\begin{aligned} b_1+b_6&=606\\ d_1+d_6&=432\\ b_1+d_1&=454\\ b_6+d_6&=584 \end{aligned}\right.$$

Hence $a_4-e_8=124$ and $b_6-d_1=752$.

There two equations are easily solved in primes, so that we have

$$e_2=17,\ a_4=141 \quad \therefore\ a_2=461,\ e_4=419$$

and
$$d_1=11,\ b_6=163 \quad \therefore\ b_1=443,\ d_6=421.$$

Hence the square is completely determined.

Question 840.

(A. C. L. WILKINSON) :—If A′, B′, C′ are the points of contact of the inscribed circle of a triangle ABC with the sides and P the Feuerbach point corresponding to the inscribed circle, show that

$$PA' : PB' : PC' = \left|\sin\frac{B-C}{2}\right| : \left|\sin\frac{C-A}{9}\right| : \left|\sin\frac{A-B}{2}\right|$$

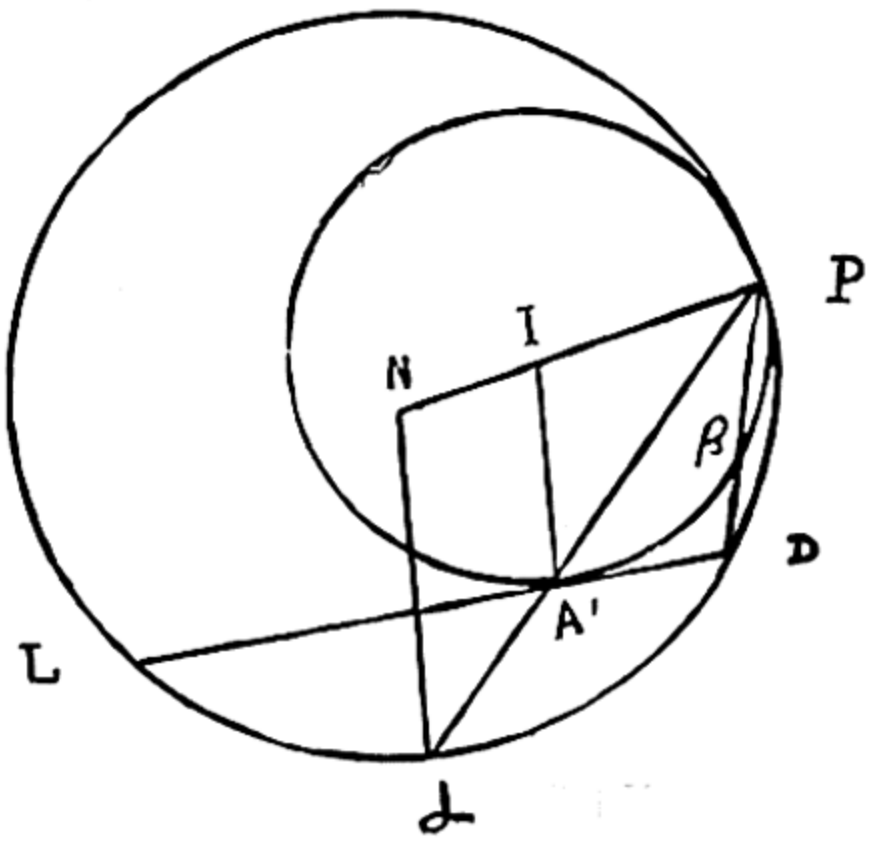
Question 849.

(K. APPUKUTTAN ERADY) :—If D, E, F be the feet of the altitudes of the triangle in the above question, shew that

$$DP : EP : FP = \operatorname{cosec}\frac{A}{2}\sin\frac{B-C}{2} : \operatorname{cosec}\frac{B}{2}\sin\frac{C-A}{2} : \operatorname{cosec}\frac{C}{2}\sin\frac{A-B}{2}.$$

Solution (1) *by S. Muthukrishnan and Zero*; (2) *by F. H. V. Gulasekharan and K. A. Erady*; (3) *by K. B. Madhava.*

(1) (i) Produce PA′ to meet the nine point circle at α, and let L, M, N be the mid-points of the sides. Then obviously



$$PA' \propto P\alpha$$

$$\therefore \quad PA'^2 \propto \alpha A'.\ \alpha P$$
$$\propto \alpha D^2.$$

$$\therefore \quad PA' \propto R \sin \tfrac{1}{2}(B-C).$$

Hence the result.

(ii) Let PD cut the incircle at β. Then

$$PD \propto P\beta$$

$$\therefore \quad PD^2 \propto D\beta.DP$$
$$\propto DA'^2.$$

$$\therefore \quad PD \propto DA'.$$
$$\propto AI \sin \tfrac{1}{2}(B-C)$$
$$\propto r\operatorname{cosec}\frac{A}{2} \sin \tfrac{1}{2}(B-C).$$

Hence the result.

(2) (i) We have

$$\frac{PA'}{A'\alpha} = \frac{r}{\rho - r}$$

where r is the in-radius and ρ the radius of the nine point circle.

Again since the chords $P\alpha$ and DL of the nine point circle intersect at A',

$$PA'.\ A'\alpha = A'D.A'L.$$

$$\therefore \quad PA'^2 = \frac{r}{\rho - r}.\ A'D.A'L.$$

$$= \frac{r}{\rho - r}.\ \frac{b-c}{2}.\ \frac{(b-c)(b+c-a)}{2a}$$

$$= \frac{32\rho^2 r}{\rho r} \sin\frac{A}{2} \sin\frac{B}{2} \sin\frac{C}{2} \sin^2\frac{B-C}{2}$$

Hence $PA' : PB' : PC' = \left|\sin\frac{B-C}{2}\right| : \left|\sin\frac{C-A}{2}\right| : \left|\sin\frac{A-B}{2}\right|$

Here $$\frac{DP}{D\beta} = \frac{\rho}{\rho - r}$$

and $$DP.\ D\beta = DA'^2.$$

$$\therefore \quad DP^2 = \frac{\rho}{\rho - r} \times DA'^2.$$

$$\therefore \quad DP : EP : FP$$
$$= DA' : EB' : FC'$$
$$= \frac{(b-c)(b+c-a)}{a} : \frac{(c-a)(c+a-b)}{b} : \frac{(a-b)(a+b-c)}{c}$$
$$= \operatorname{cosec}\frac{A}{2} \sin\frac{B-C}{2} : \operatorname{cosec}\frac{B}{2} \sin\frac{C-A}{2} : \operatorname{cosec}\frac{C}{2} \sin\frac{A-B}{2}.$$

(3)

(i) The areal co-ordinates of the Feuerbach point P with respect to the triangle A′B′C′ are (cf: J. I. M. S., Vol ix, No. 2 § 6),

$$\frac{a}{b-c}\,;\ \frac{b}{c-a}\,;\ \frac{c}{a-b}$$

i.e. $\cos\frac{A}{2}\operatorname{cosec}\frac{B-C}{2}\,;\ \cos\frac{B}{2}\operatorname{cosec}\frac{C-A}{2}\,;\ \cos\frac{C}{2}\operatorname{cosec}\frac{A-B}{2}$.

They are also

$$\frac{\sin\ B'\hat{A}'C'}{P\,A'}\,;\ \frac{\sin A'\hat{B}'C'}{P\,B'}\,;\ \frac{\sin A'\hat{C}'B'}{P\,C'}$$

that is $\dfrac{\cos\frac{A}{2}}{P\,A'}\,;\ \dfrac{\cos\frac{B}{2}}{P\,B'}\,;\ \dfrac{\cos\frac{C}{2}}{P\,C'}$.

Hence we have at once

$$P\,A' : P\,B' : P\,C' = \sin\frac{B-C}{2} :\ \sin\frac{C-A}{2} :\ \sin\frac{A-B}{2}.$$

(ii) The areal co-ordinates of P *w. r. t.* the △DEF (cf. J.I.M.S. vol. ix p. 66).

$$\frac{a(b-c)\cos A}{\sin^2\frac{B-C}{2}}\,;\ \frac{b(c-a)\cos B}{\sin^2\frac{C-A}{2}}\,;\ \frac{c(a-b)\cos C}{\sin^2\frac{A-B}{2}}$$

i.e. $\dfrac{\sin 2A\sin\frac{A}{2}}{\sin\frac{B-C}{2}}\,;\ \dfrac{\sin 2B\sin\frac{B}{2}}{\sin\frac{C-A}{2}}\,;\ \dfrac{\sin 2C\sin\frac{C}{2}}{\sin\frac{A-B}{2}}$

They are also

$$\frac{\sin EDF}{DP}\,,\ \frac{\sin DEF}{EP}\,,\ \frac{\sin DFE}{FP}$$

i.e. $\dfrac{\sin 2A}{DP}\,,\ \dfrac{\sin 2B}{EP}\,,\ \dfrac{\sin 2C}{FP}$

Hence we have at once

$$DP : EP : FP :$$
$$= \operatorname{cosec}\frac{A}{2}\sin\frac{B-C}{2} :\ \operatorname{cosec}\frac{B}{2}\sin\frac{C-A}{2} :\ \operatorname{cosec}\frac{C}{2}\sin\frac{A-B}{2}.$$

Solution of Q. 840 *by Sadanand, R. D. Karve and K. R. Rama Iyer.*

If θ be the angle A′P subtends at I

$$\left(\frac{R}{2}-r\right)\cos\theta = \frac{1}{2}(R\cos A + 2R\cos B\cos C) - r$$
$$= \left(\frac{R}{2}-r\right) - R\sin^2\left(\frac{B-C}{2}\right).$$

$$\therefore\quad \left(\frac{R}{2}-r\right)\sin\frac{\theta}{2} = R\sin^2\frac{B-C}{2}$$

or $$\left| \sin\frac{\theta}{2} \right| = \left(\frac{R}{R-2r}\right)^{\frac{1}{2}} \left| \sin\frac{B-C}{2} \right|$$

$$\therefore \quad A'P = 2r\sin\frac{\theta}{2} = 2r\left(\frac{R}{R-2r}\right)^{\frac{1}{2}} \left| \sin\left(\frac{B-C}{2}\right) \right|$$

$$\therefore \quad A'P : B'P : C'P$$

$$= \left| \sin\frac{B-C}{2} \right| : \left| \sin\frac{C-A}{2} \right| : \left| \sin\frac{A-B}{2} \right|$$

Additional Solutions by M. K. Kewalramani, Hem Raj, N. P. Pandya and S. V. Venkatachalayyar.

Question 855.

(Martyn M. Thomas, M A.):—An ellipse is inverted with respect to any point on its circumference. Show that the inverse curve has three and only three points of inflexion, and that these are collinear.

Solution (1) *by K. B. Madhava,* (2) *by Sadanand and K. R. Rama Aiyer.*

(1) The equation of an ellipse referred to a point on the circumference can be put in the form

$$ax^2+2hxy+by^2+2gx+2fy=0;$$

and its inverse *w.r.t* a circle of radius $=k$ can be easily seen to be the cubic,

$$2(gx+fy)(x^2+y^2)+k^2(ax^2+2hxy+by^2)=0,$$

(*Cf*: Edwards: Diff. Cal. p. 173 Ex. 2).

This form shows at once that the origin is a node with the nodal tangents

$$ax^2+2hxy+by^2=0.$$

Hence by Plücker's second equation (Edwards p. 255), the number of inflexional points $=9-6\delta=3$; and only three; and it is well-known that the three points of inflexion of a cubic are collinear.

(2) Let the ellipse be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

Let P be any point on the ellipse and let α be its eccentic angle. Then we know that there are three and only three points on the ellipse whose circles of curvature pass through P and that their eccentric angles are $\frac{1}{3}(2\pi-\alpha)$, $\frac{1}{3}(4\pi-\alpha)$ and $\frac{1}{3}(6\pi-\alpha)$. If Q be one of these points then its circle of curvature cuts the ellipse in three adjacent points. If we invert the ellipse *w. r. t.* P the circle of curvature inverts into a

straight line which cuts the inverse curve in three adjacent points. Therefore Q inverts into a point of inflexion. Similarly the other two points invert into two points of inflexion. Hence the inverse curve has three and only three points of inflexion. Also P and the three points on the ellipse form a cyclic quadrilateral because the sum of their eccentric angles $= \alpha + \frac{1}{3}\{2\pi - \alpha + 4\pi - \alpha + 6\pi - \alpha\} = 4\pi$.

The circumcircle of this cyclic quadrilateral inverts into a straight line passing through the three points of inflexion in the inverse curve. Hence the three points of inflexion are collinear.

Question 881.

(S. KRISHNASWAMI AIYANGAR) :—Prove that

$$(i)\int_0^{\frac{\pi}{2}} \frac{(\log \sin x)^2}{\sqrt{\sin x}}dx = \frac{\{\Gamma(\frac{1}{4})\}^2(\pi^2+16)}{8\sqrt{2\pi}}$$

$$(ii)\int_0^{\frac{\pi}{2}} \frac{(\log \sin x)^3}{\sqrt{\sin x}}dx = \frac{\{\Gamma(\frac{1}{4})\}^2(\pi^2-32\pi-112)}{16\sqrt{2\pi}}$$

Remarks by K. B. Madhava and Sadanand.

The first part of the question is completely solved in Bromwich, Ex. 45 p. 476, taking $\alpha = \frac{1}{4}$; second part follows from the same, by differentiation. It appears the right hand sides of these integrals are wrongly given.

Question 892.

(S. KRISHNASWAMI IYENGAR) :—Show that

$$\int_0^{\frac{\pi}{2}} \sqrt[3]{\sin x \log \sin x}dx = \frac{\pi \sqrt[3]{2.\sqrt{3}}}{\Gamma(\frac{1}{3})}\left\{\frac{\pi}{3}\sqrt{3} + \log 2 - 3\right\}.$$

Question 893.

(S. KRISHNASWAMI IYENGAR) :—Find the value of

$$\int_0^{\frac{\pi}{2}} \frac{\log \sin x . \log \cos x dx}{\sqrt{\sin x \cos x}}.$$

Solution by R. J. Pocock, S. R. Ranganathan and K. B. Madhava.

We have

$$\int_0^{\frac{\pi}{2}} \sin^{2\alpha-1} x.\ \log\ \sin x dx = \frac{\sqrt{\pi}}{4}.\ \frac{\Gamma(\alpha)}{\Gamma(\alpha+\frac{1}{2}\delta)}\left[\psi(\alpha)-\delta(\alpha+\tfrac{1}{2}\psi)\right]$$

(Bromwich p. 476 Ex. 45.)

Write $\alpha=\frac{2}{3}$. Now $\psi\left(\frac{2}{3}\right)=-c-\frac{3}{2}\log 3+\frac{2}{6}\pi\sqrt{3}$

(loc. cit. Ex. 43.)

Also $$\psi\left(\frac{2}{3}\right)+\psi\left(\frac{7}{6}\right)=2\psi\left(\frac{4}{3}\right)-2\log 2$$

and $$2\psi\left(\frac{4}{3}\right)-2\psi\left(\frac{1}{3}\right)=6.$$

(loc. cit. Ex. 42.)

Hence $$\psi\left(\frac{7}{6}\right)=-c+6-2\log 2-\frac{3}{2}\log 3-\pi\frac{\sqrt{3}}{2}.$$

$$\therefore \psi\left(\frac{2}{3}\right)-\psi\left(\frac{7}{6}\right)=\frac{2}{3}\pi\sqrt{3}+2\log 2-6.$$

Also $$\frac{2^{2x}.\Gamma(x+\frac{1}{2}).\Gamma(x+1)}{\Gamma(2x+1)}=\sqrt{\pi}$$

(Bromwich p. 461.)

$$\therefore \quad \Gamma\left(\frac{7}{6}\right).\Gamma\left(\frac{2}{3}\right)=\frac{3}{2}.\frac{\sqrt{\pi}}{2^{\frac{1}{3}}}.\Gamma\left(\frac{1}{3}\right). \quad \text{Also } \Gamma\left(\frac{2}{3}\right)=\frac{2\pi}{\sqrt{3}\Gamma(\frac{1}{3})}.$$

$$\therefore \quad \int_0^{\frac{\pi}{2}} \sqrt[2]{\sin x.\log\sin x dx}=\frac{\sqrt{\pi}}{2}.\frac{\Gamma(\frac{2}{3})}{\Gamma(\frac{7}{6})}.\left(\frac{\pi\sqrt{3}}{3}+\log 2-3\right)$$

$$=\frac{\sqrt[3]{2.\pi^2}}{\{\Gamma(\frac{1}{3})\}^3}=\left\{\frac{\pi\sqrt{3}}{3}+\log 2-3\right\}.$$

[*N.B.*—The result as printed appears to be inaccurate.].

Again,

$$\int_0^{\frac{\pi}{2}} \frac{\log\ \sin x\ \log\ \cos x}{\sqrt{\sin x\cos x}},$$

$$=\frac{1}{4}\left[\frac{d^2}{d\alpha d\beta}\int_0^{\frac{\pi}{2}} \sin^{2\alpha+1} x\cos^{2\beta+1} x\,dx\right]\alpha=\beta=\tfrac{1}{4}$$

$$=\frac{1}{8}\left[\frac{d^2}{d\alpha d\beta}\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}\right]\alpha=\beta=\tfrac{1}{4}$$
$$=\frac{1}{8}\frac{\{\Gamma(\frac{1}{4})\}^2}{\Gamma(\frac{1}{2})}\left[\{\psi(\tfrac{1}{4})-\psi(\tfrac{1}{2})\}^2-\psi'(\tfrac{1}{2})\right]$$
$$=\frac{1}{8}\frac{\{\Gamma(\frac{1}{4})\}^2}{\sqrt{\pi}}\left[\left(\log 2+\frac{\pi}{2}\right)^2-\frac{\pi^2}{2}\right]$$

[For values of $\psi(\frac{1}{4})$, $\psi(\frac{1}{2})$, $\psi'(\frac{1}{2})$, see Bromwich: *Infinite Series*, Appendix III, Exs. 42 and 43.]

Question 911.

(R. J. Pocock) :—Show that the locus of a point such that the eccentric angles of the feet of the four normals which can be drawn from it to the ellipse $x^2/a^2+y^2/b^2=1$ satisfy the relations

$$\Sigma \cos\phi_1 \cos\phi_2=\Sigma \sin\phi_1 \sin\phi_2=0$$

is the ellipse

$$x^2+(1-e^2)y^2=a^2e^4.$$

Solution by K. J. Sanjana, H. R. Kapadia and K. B. Madhava.

The normal at the point of eccentric angle ϕ is

$$ax \sec\phi-by \operatorname{cosec}\phi=a^2-b^2=c^2;$$

if this goes through (h, k) we have $ah \sec\phi-bk \operatorname{cosec}\phi=c^2$.

Replacing by $\cos\phi$ and $\sin\phi$ and finding each in turn by rationalising, this condition can be written in the form—

$$c^4\cos^4\phi-2c^2ah\cos^3\phi+(a^2h^2+b^2k^2-c^4)\cos^2\phi+2c^2ah\cos\phi-a^2h^2=0,$$
$$c^4\sin^4\phi+2c^2bk\sin^3\phi+(a^2h^2+b^2k^2-c^4)\sin^2\phi-2c^2bk\sin\phi-b^2k^2=0.$$

Both the given relations give rise to the same result, *viz.*

$$a^2h^2+b^2k^2-c^4=0.$$

Thus the locus of the intersection of the normals is

$$x^2+\frac{b^2}{a^2}a^2=y^2\left(\frac{c^2}{a^2}\right)^2,$$

which is the result in the question. [See J. I. M. S., Vol. VIII, p. 233.]

Additional Solution by R. Srinivasan.

Question 914.

(Prof. K. J. Sanjana) :—The sum of the $2n+1$ terms of the type $\frac{\alpha}{\alpha^3}$ where α is a $(2n+1)^{th}$ root of unity is $(-)^n(n+\frac{1}{2})$.

Solution by K. B. Madhava.

Since α satisfies $x^{2n+1}-1=0$,

$\frac{1}{1\pm i\alpha}$ satisfy $\pm i^{2n+1}(z-1)^{2n+1}+z^{2n+1}=0$ respectively.

Hence,

$$\sum\frac{\alpha}{1+\alpha^2}=\frac{1}{2i}\sum\left[\frac{1}{1-i\alpha}-\frac{1}{1+i\alpha}\right]$$

$$=\frac{1}{2i}\left[\frac{(2n+1)i^{2n+1}}{1+i^{2n+1}}+\frac{(2n+1)i^{2n+1}}{1-i^{2n+1}}\right]$$

$$=\frac{1}{2i}\times(2n+1)i_2^{n+1}=(-)^n(n+\tfrac{1}{2}).$$

Slightly different solutions by S. R. Ranganathan and R. Srinivasan.

Question 917.

(HEM RAJ) :—In any triangle shew that

$$2\Sigma a^2(b+c)\geq\Sigma(a^3)+9abc.$$

Solution by K. J. Sanjana.

It may be shown that if G is the centroid and I the incentre,

$$GI^2+4Rr=\tfrac{1}{3}(bc+ca+ab)-\tfrac{1}{9}(a^2+b^2+c^2).$$

[see Hobson, *Plane Trio* , p. 200.]

Hence, unless the triangle is equilateral in which case the inequality becomes an equality, we get

$$\tfrac{1}{3}(bc+ca+ab)>\tfrac{1}{9}(a^2+b^2+c^2)+4Rr$$

i.e.
$$>\tfrac{1}{9}(a^2+b^2+c^2)+\frac{2abc}{a+b+c};$$

so that $3\,(a+b+c)(bc+ca+ab)>(a+b+c)(a^2+b^2+c^2)+18abc,$

or $3\Sigma a^2(b+c)+9abc>\Sigma a^2(b+c)+\Sigma(a^3)+18abc,$

i.e. $2\Sigma a^2(b+c)>\Sigma(a^3)+9abc.$

QUESTIONS FOR SOLUTION.

927. (M. K. KEWALRAMANI) :—ABCDE . is a polygon whose sides taken in order are a, b, c, d, e ... If the angles which the sides b, c, d, e...make with the positive direction of the side a, be respectively called ab, ac, ad, ae ..., prove that

$$a^n = \sum \left[\frac{n!}{r!\, s!\, t! \ldots} b^r c^s d^t \ldots\ldots (\cos \{ r.ab + s.ac + t.ad + \ldots \}) \right]$$

were n is a positive integer and r, s, t, ... are also positive integers including zero, but such that $r+s+t+\ldots = n$ and Σ denoting summation of like expressions when, r, s, t, ... are taken in their entirety subject to the above restriction ; of course on the right hand side, a not occurring.

928. (M. K. KEWALRAMANI) :—Show that the infinite product

$$\left(1-\frac{1}{2^{12}}\right)\left(1-\frac{1}{3^{12}}\right)\left(1-\frac{1}{4^{12}}\right)\ldots$$

$$= \frac{\sinh \pi \cosh^2 \frac{\sqrt{3}\pi}{2}}{24\pi^6} [\cosh \pi - \cos \sqrt{3}\pi]$$

929. (S. MALHARI RAO) :—Express 2595600 as the product of two factors such that the sum of the factors of either factor may be equal to the other factor.

930. (S. MALHARI RAO) :—Fill up the vacant cells in the following figure with the remaining numbers of the series 1, 2, 3, ..., 63, 64, so that the whole figure and each of the four corner sub-squares of 16 cells, may all be pan-diagonal magic squares.

39	28	38	25				
24	29	35	32				
27	40	26	37				
30	33	31	36				

931. (C. KRISHNAMACHARY):—Sum to n terms and discuss the convergence to infinity of the series.

$$\frac{y}{x+y}+x\frac{y+z}{(x+y)(x+y+z)}+x^2\frac{y+2z}{(x+y)(x+y+z)(x+y+2z)}+\ldots$$

932. (T. KRISHNA RAO) :—In Q. 772 prove the following construction :—Take MH equal to the given sum of the sides and bisect it at F. Draw the angle FMK equal to the vertical angle. Cut off MK such that the rect. MF. MK = the given rectangle. Join FK and draw the bisector of the $\angle$MFK. On the bisector cut off FG such that $FG^2 = FM.FK$. Join GM and GH. From GM cut off GL = GH. Let the bisector of $\angle$MGH meet MH in N. Join LN. Then $\triangle$ LMN is the required triangle.

933. (K. B. MADHAVA):—With the usual notation of the Elliptic Functions, show that

$$\int_0^{\frac{1}{\sqrt{2}}} k\,KK'\,dk=2^6\int_0^{\frac{1}{\sqrt{2}}} k\,EE'\,dk=\frac{\pi}{32}^3.$$

934. (HEMRAJ):—If n be a positive integer, show that

$$\frac{1\cdot3\cdot5\ldots\ldots(2n-1)}{2^{n-1}}\left\{n+\frac{n(n-1)}{1\cdot2}\left(1-\tfrac{1}{3}\right)+\frac{n(n-1)(n-2)}{1\cdot2\cdot3}\left(1-\tfrac{1}{3}+\tfrac{1}{5}\right)+\ldots\right.$$
$$\ldots+\frac{n(n-1)}{1\cdot2}\left(1-\tfrac{1}{3}+\tfrac{1}{5}-\ldots\ldots+(-1)^{n-2}\frac{1}{2n-5}\right)$$
$$+n\left(1-\tfrac{1}{3}+\tfrac{1}{5}-\ldots\ldots+(-1)^{n-2}\frac{1}{2n-3}\right)$$
$$\left.+\left(1-\tfrac{1}{3}+\tfrac{1}{5}-\ldots\ldots+(-1)^{n-1}\frac{1}{2n-1}\right)\right.$$

is an integer.

935. (K. J. SANJANA, M.A.) :—Let a, b, c, be numbers representing the sides of a real triangle, and

$$S\equiv a^2(b+c)+b^2(c+a)+c^2(a+b),$$
$$\Sigma\equiv a^3+b^3+c^3,\ P\equiv abc:$$

prove that $6S > \text{or} < l\Sigma+(36-3l)\,P,$

according as $l=0, 1, 2, 3$ or $=6, 7, 8, 9, 10, 11, 12.$

Also examine the inequality for $l=4$ and $l=5$.

936. (K. J. SANJANA, M. A.) :—If [Suggested by Q. 917.]

$$f(x)=x^6+55x^4+439x^3+225,$$

shew that

$$\sum_1^\infty\frac{(-)^{m-1}}{2mf(2m-1).f(2m+1)}=\frac{\log 4-1\cdot23}{2073600},$$

(Suggested by Q. 908.)

937. (H. R. KAPADIA):—If n have any positive integral value except unity, and r be any positive integer which is not a perfect power, show that

$$\Sigma(n-1)/(r^n-1)=\frac{\pi^2}{6};$$

and if $d(n)$ denote the number of divisors of n, that $\Sigma(d(n)-1)/r^n=1$; also that $\Sigma(n-1)/r=\Sigma 1/(r-1)^2$.

938 (H. R. KAPADIA):—Prove that, if $m_1, m_2, \ldots\ldots m_s$ are the integers less than and prime to m, and if $\rho_1, \rho_2, \ldots\ldots$ are the different prime factors of m,

$$\prod_1^s \sin\left(\theta+\frac{m_r\pi}{m}\right)=\frac{\sin m\theta.\prod\sin\frac{m\theta}{p_1p_2}.\prod\sin\frac{m\theta}{p_1p_2p_3p_4}\ldots\ldots}{2^s\prod\sin\frac{m\theta}{p_1}.\prod\sin\frac{m\theta}{p_1p_2p_3}}$$

939. (S. R. RANGANATHAN):—If ρ,ρ_0 and ρ_α are the radii of curvature, at the corresponding points of a curve, its evolute, and its α—evolute respectively, show that as α varies from $-\frac{\pi}{2}$ to $+\frac{\pi}{2}$, ρ_α has for its maximum value the expression $\sqrt{\rho^2+\rho_0^2}$ and that it attains it when $\tan\alpha=\rho/\rho_0$.

940 (R. SRINIVASAN, M.A., M.R.A.S.):—Show that

$$\int_0^1 \frac{x^n\log x}{1+x^2}dx=\frac{\pi^2}{4}\sec\frac{n\pi}{2}\tan\frac{n\pi}{2};$$

941. BALAK RAM):—[Suggested by Q. 817].

Solve $$x!=(x-y)!\ (y!)\ PQR$$

where x and y may be any integers satisfying

$$3\angle y\frac{x}{2},$$

and P, Q, R are primes.

922. (HEMRAJ):—Show that if p be prime,

$$\left[\left(\sum_1^{p-1}\frac{1}{n}\right)\div\frac{p3}{p!}\right]$$

is an integer.

List of Periodicals Received.

(*From* 16*th November* 1917 *to* 15*th January* 1918.)

1. Acta Mathematica, Vol. 41, No. 2.
2. American Mathematical Monthly, Vol. 24, No. 7, September 1917.
3. Astrophysical Journal, Vol. 46, No. 2, September 1917.
4. Bulletin of the American Mathematical Society, Vol. 24, No. 1, October 1917.
5. Mathematical Gazette, Vol. 9, No. 130, July 1917. (3 Copies).
6. Mathematical Questions and Solutions, Vol. 4, Nos. 3, 4 and 5, September, October and November 1917. (5 Copies).
7. Mathematical Teacher, Vol. 10, No. 1, September 1917.
8. Messenger of Mathematics, Vol. 46, No. 12, April 1917 and Vol. 47, No. 1, May 1917.
9. Philosophical Magazine, Vol. 34, Nos. 202 and 203, October and November 1917.
10. Popular Astronomy, Vol. 25, No. 8, October 1917. (3 Copies).
11. Proceedings of the Royal Society of London, Vol. 93, No. 655, October 1917.
12. Quarterly Journal of Mathematics, Vol. 48, No. 1, October 1917.
13. School Science and Mathematics, Vol. 17, No. 7, October 1917.
14. Tohoku Mathematical Journal, Vol. 12, No. 3, October 1917.
15. Transactions of the Royal Society of London, Vol. 217, No. 558.

Complex Roots of Equations.

By M. T. Naraniengar.

Introduction.

It is proposed to discuss in this paper *graphic methods* of visualising the complex roots of an equation. To this end Argand diagrams are freely used, and complex roots obtained as the *real* intersections of two plane curves. The method of the Theory of Functions of Complex Variables is briefly indicated towards the close of the paper.

The paper is conveniently divided into three Sections: (1) The First Section treats of equations up to the fifth degree by means of the elementary methods of curve tracing; (2) In the second Section, Approximations to Complex Roots of a general algebraical equation, are developed by means of Cauchy's expansion and Taylor's theorem, and *incidentally* a "Method of dealing with the Intersections of Plane Curves" is referred to and its connection with the properties of Polar Curves explained; (3) The third and last Section is devoted to a brief discussion of Transcendental equations.

The Post Script deals with simultaneous equations involving two Complex Variables.

The computation of imaginary roots of Numerical Algebraic Equations has not received much attention on the part of mathematicians. There is a reference to a modification of Horner's method so as to apply to imaginary roots on page 12 of Mathematical Monographs, No. 10.—*The Solution of Equations* by M. Merriman*. Mc Clintock in 1894 published a method of development in series of the roots of an equation by means of his Calculus of Enlargement†. His method can be used for approximate computations when the series is convergent. Lambert in 1903 gave out a similar method of expansion by Maclaurin's formula, applicable to Algebraic, as well as, Transcendental Equations‡. This method consists in introducing a second variable into all the terms but two of an equation and putting it equal to unity after the expansion of the first variable is obtained by Maclaurin's theorem.

Dr. F. S. Macaulay in his *Algebraic Theory of Modular Systems*, 1916, No. 19 of Cambridge Tracts, gives the following further references: Transactions of the American Mathematical Society, No. 3, 1902; and No. 5, 1904.

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* Sheffler, Die Auflösung der algebraischen and transzendenten Gleichungen Braunschweig, 1859; and Jelink, Die Auflosung der hoheren numerischen Gleichungen, Leipzig, 1865.

† Bulletin of American Mathematical Soc., 1894, Vol. I, p. 37; also, American Journal of Mathematics, Vol XVII, pp. 89—110.

‡ American Philosophical Society Proceedings, Vol. 42.

Section (1).

1. Consider the quadratic equation

$$z^2+2pz+q=0. \qquad (1)$$

We may write its roots in the form $(x+iy)$ without loss of generality. Substituting in the equation and separating real and imaginary parts, we have

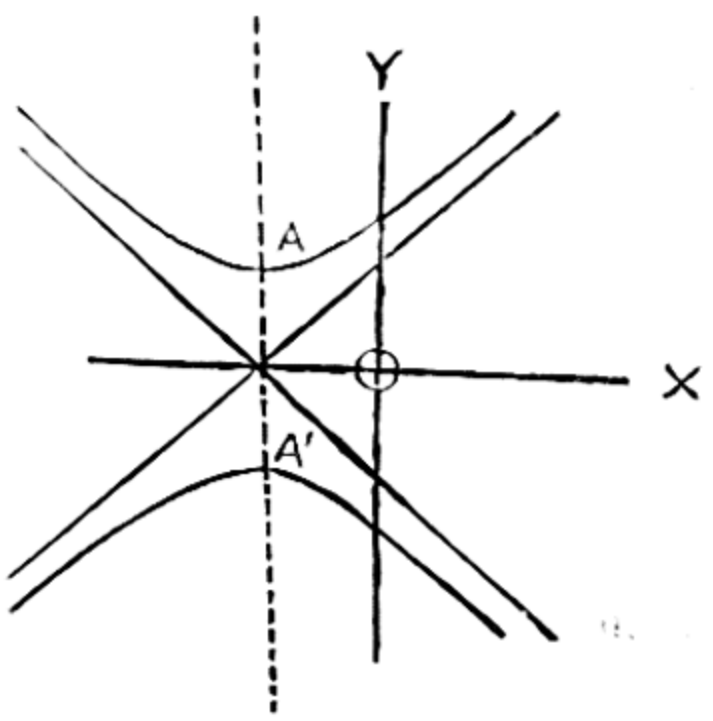
$$(x^2-y^2+2px+q)+2iy(x+p)=0,$$

which is equivalent to the two equations

$$\left.\begin{aligned}(x^2-y^2+2px+q=0.\\ y(x+p)=0.\end{aligned}\right\} \qquad (2)$$

Drawing the curves (2), we find (x,y) as their intersection. In other words, the Argand representations of the complex roots $(x+iy)$ are the *real* intersections of the Cartesian curves (2). It is quite easy to see that these intersect outside the axis of x only when the roots of

Fig. 1



the quadratic are complex. In this case the graphic points A, A′ will represent the complex roots

$$\{-p\pm i\sqrt{q-p^2}\} \qquad \text{[see Fig. 1.]}$$

2. Next, suppose the equation is

$$f(z)\equiv z^3+3pz^2+3qz+r=0. \qquad \ldots \qquad \ldots \qquad (3)$$

Proceeding as before, we have

$$f(x\pm iy)=f(x)\pm iy\,f'(x)-\frac{y^2}{2}f''(x)\mp\frac{iy^3}{6}f'''(x)=0,$$

which breaks up into the two separate equations

$$\left.\begin{aligned}f(x)-\tfrac{1}{2}\,y^2\,f''(x)=0\\ y[f'(x)-\tfrac{1}{6}\,y^2\,f'''(x)]=0\end{aligned}\right\}$$

The factor $y=0$, in the second equation leads to $f(x)=0$, determining the real roots of the proposed equation. The complex roots (when they exist) are represented graphically as the real intersections of

$$\left.\begin{aligned}f(x)-\tfrac{1}{2}\,y^2\,f''(x)=0\\ f'(x)-\tfrac{1}{6}\,y^2\,f'''(x)=0.\end{aligned}\right\} \qquad \ldots \qquad \ldots \qquad (4)$$

Now the cubic and conic represented by (4) can be readily traced and their intersections obtained; or (4) may be reduced to a cubic equation in x by eliminating y^2 and the real roots of this resulting cubic obtained. The corresponding real values of y will then give the graphic points, required.

[*cf.* MacRobert's *Theory of Functions*, pp. 16—19.]

3. The biquadratic equation $f(z)\equiv 0$, treated similarly leads to the intersections of the curves

$$\left.\begin{aligned} f(x)-\tfrac{1}{2}y^2 f''(x)+y^4=0,\\ f'(x)-\tfrac{1}{6}y^2 f'''(x)=0,\end{aligned}\right\} \quad \ldots \quad \ldots \quad (5)$$

which are of the 4th and 3rd degrees respectively.

The elimination of y^2 between these gives an equation in x which may also be used for locating the real values of x, but the process is tedious.

4. Instead of the Cartesian $(x+iy)$ for the complex root, we may take the standard form $r(\cos\theta \pm i\sin\theta)$. The auxiliary curves corresponding to the complex roots will then reduce to

$$r^n\cos n\theta+p_1r^{n-1}\cos(n-1)\theta+p_2r^{n-2}\cos(n-2)\theta+\ldots+p_n=0,$$
$$r^n\sin n\theta+p_1r^{n-1}\sin(n-1)\theta+p_2r^{n-2}\sin(n-2)\theta+\ldots+p_n=0.$$

5. In connection with the method of this section the following hints on *curve-tracing* may be of use:

Suppose $F(x,y)=0$ is any plane curve. Then in relation to the curve the set of points in the plane of the curve may be grouped into three classes:

(i) The group of points for which $F(x,y)$ takes a positive value; say, the *positive group*.

(ii) The group of points for which $F(x,y)$ takes a negative value; say, the *negative group*.

(iii) The group of points for which $F(x,y)$ takes a zero value; say the *zero group*.

The last group defines the curve, and the other two groups are separated by it. Thus the whole plane is divided into *compartments* of *positive* and *negative* points, whose boundary is the curve itself.

Now, let P be any positive point and Q any negative point. Then by the principle of continuity we infer that there must be some point between P and Q lying on the curve, which can be approximately

found. The method, though laborious, is useful in locating a curve consisting of several branches. Further, its essential importance as a method consists in its negative character; it does not require any knowledge of points on the curve, as in the usual method of curve tracing. By this method we can locate a curve of any degree in relation to two points taken at random.

Ex. 1.—As an illustration, let us consider the equation $x^5-4x-2=0$ [Cajori: *Theory of Equations*, p. 29].

The auxiliary curves being denoted by P and Q, we have

$P\equiv r^5\cos 5\theta-4r\cos\theta-2\equiv x^5-10x^3y^2+5xy^4-4x-2=0,$

$Q\equiv r^5\sin 5\theta-4r\sin\theta\equiv y(5x^4-10x^2y^2+y^4-4)=0.$

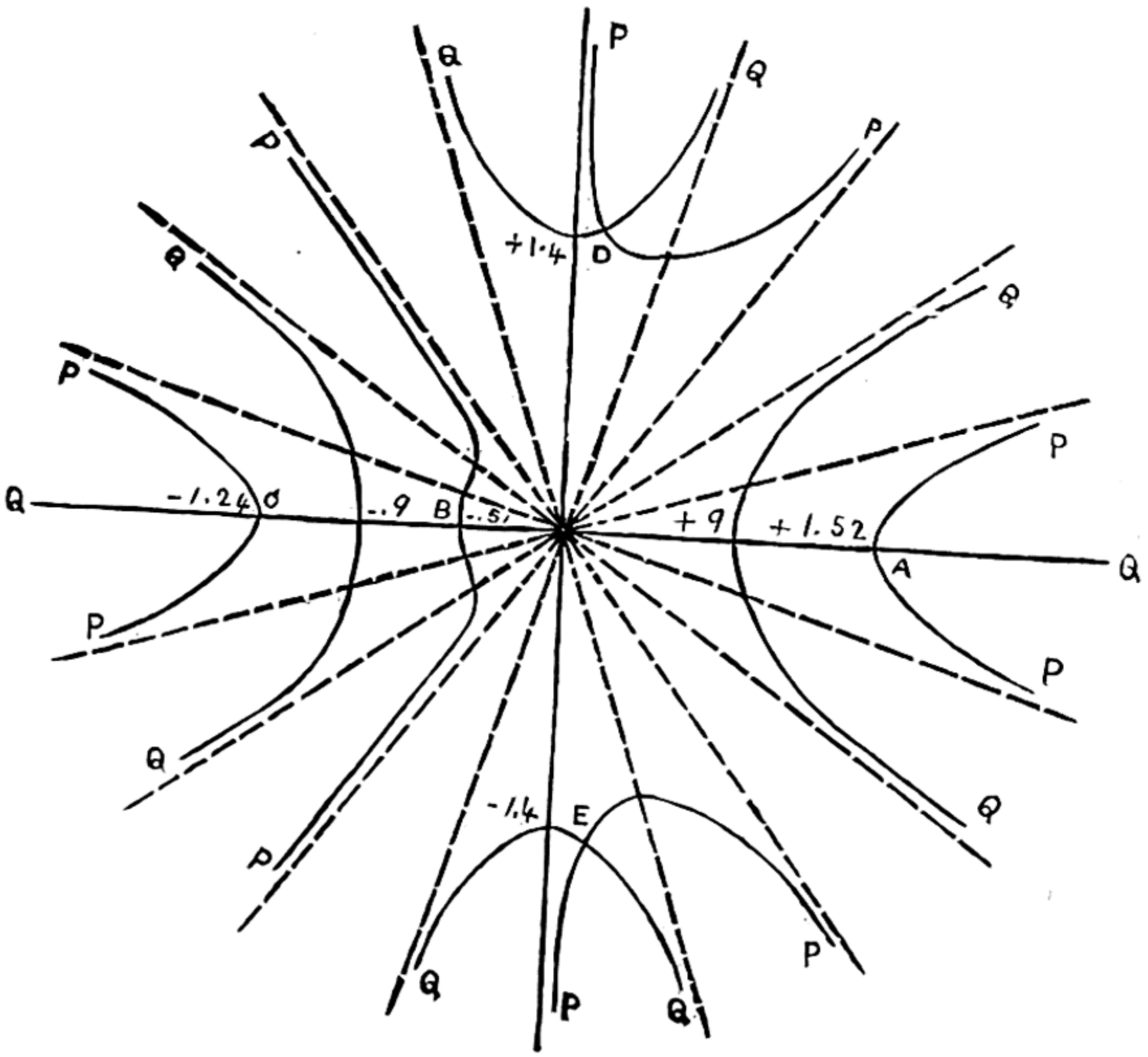
(I) *The P—curve*:—Considering P as a quadratic in y^2, we find that the discriminant

$D=100x^6-20x(x^5-4x-2)=40(2x^6+2x^2+x)$ and is positive for all values of x not lying between 0 and $-\cdot48$, since $D=0$ for $x=-\cdot48$ nearly.

Hence y has real values except when x lies between these limits.

Again all the values of y will be real only, if $\phi\equiv(x^5-4x-2)/x$ is positive. The latter changes sign when $x=1\cdot52,-\cdot51,-1\cdot24$ nearly.

Fig. 2.



Thus the following cases arise

(i) $\phi = +$, when $x > 1{\cdot}52$;
(ii) $\phi = -$, when $1{\cdot}52 > x > 0$;
(iii) $\phi = +$, when $0 > x > -{\cdot}51$;
(iv) $\phi = -$, when $-{\cdot}51 > x > -1{\cdot}24$;
(v) $\phi = +$, when $-1{\cdot}24 > x$.

In case (i) y has *four real* values,
(ii) y has *two real* and *two imaginary* values,
(iii) y has *four imaginary* values, if $0 > x > -{\cdot}48$, and *four real* values, if $-{\cdot}48 > x > -{\cdot}51$,
(iv) y has *two real* and *two imaginary* values,
(v) y has *four real* values.

(2) *The Q—Curve* :—This consists of the axis of x and two hyperbolic branches as is easily seen.

[*N.B.*—The asymptotes to the P and Q curves are $\cos 5\theta = 0$, $\sin 5\theta = 0$ respectively].

A tracing of the curves is given in Fig. 2,

Their *real intersections* furnish the roots of the quintic, and they are

A, B, C, corresponding to the real roots;
D, E, corresponding to the imaginary roots.

Their approximations are

$$+1{\cdot}52,\ -{\cdot}51,\ -1{\cdot}24;\ +{\cdot}12 \pm 1{\cdot}44\sqrt{-1}$$

The approximation to D is found by changing the origin to $(0, \sqrt{2})$, so that Q comes

$$y' = \frac{5\sqrt{2}}{4} x'^2, \text{ nearly};$$

and P becomes (on substitution for y)

$$50x'^3 + gx' - 1 = 0,$$

whence
$$x' = \tfrac{1}{8} = {\cdot}12$$
$$y' = \frac{5\sqrt{2}}{4} \cdot \frac{1}{64} = {\cdot}03$$

or
$$x = x' = .12!\ ;\ y = \sqrt{2} + y' = 1{\cdot}44.$$

It may be remarked here, that the P and Q curves are orthogonal always; and that the P—curve cuts the axis of x in $f(x) = 0$, and the Q—curve in $f'(x) = 0$.

By Rolle's theorem, therefore, it follows that if P crosses the axis p times, Q crosses it in *at least* $(p-1)$ times. If all the roots of $f(z)$ are real, P and Q will never cut in real points.

Ex. 2. As a second example, we may take the cubic

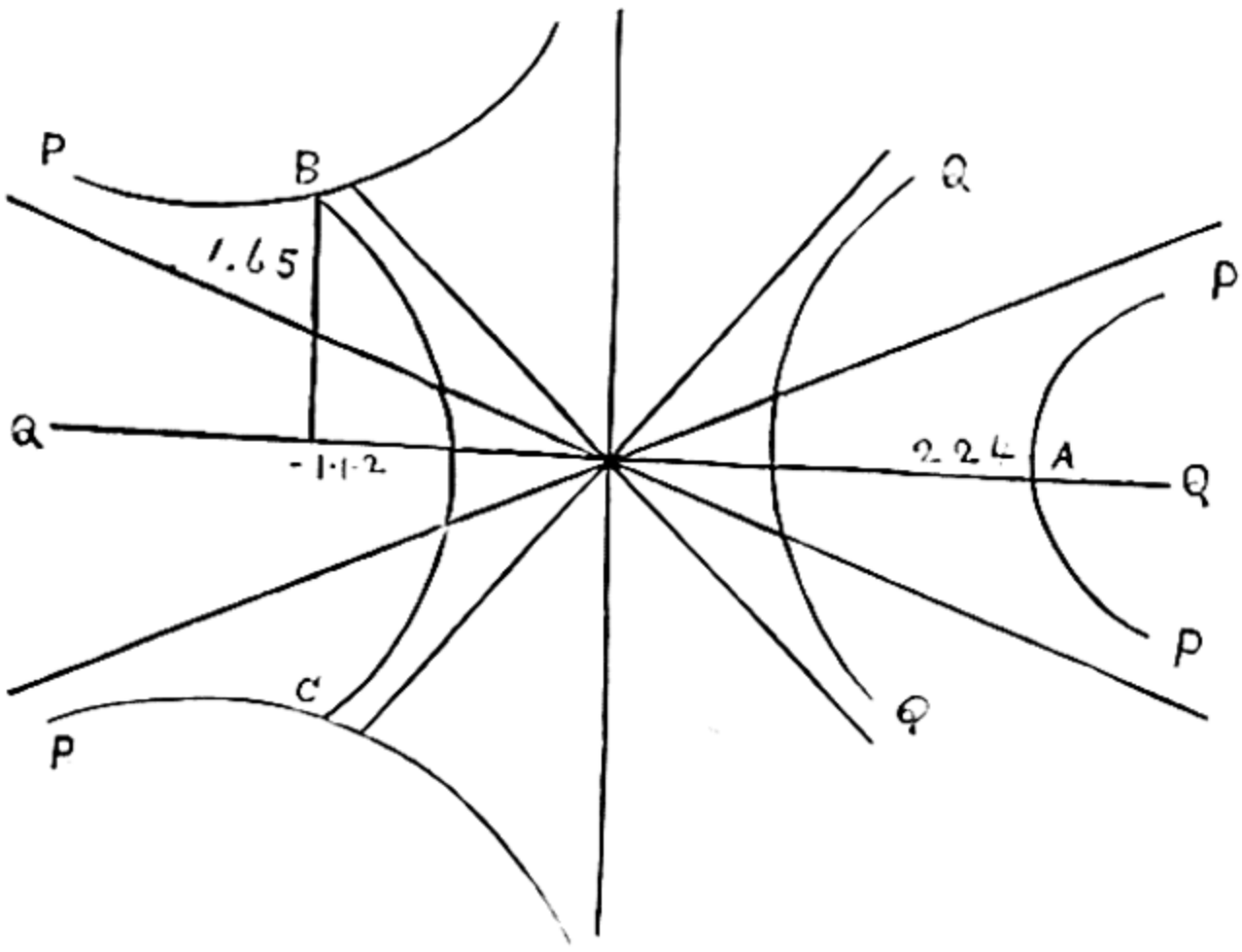
$$z^3 - z - 9 = 0.$$

Here $P \equiv x^3 - 3xy^2 - x - 9 = 0$,

$Q \equiv y(3x^2 - y^2 - 1) = 0$.

The graphs are as under.

Fig. 3.



The roots are $+2{\cdot}24$; $-1{\cdot}12 \pm 1{\cdot}65\sqrt{-1}$.

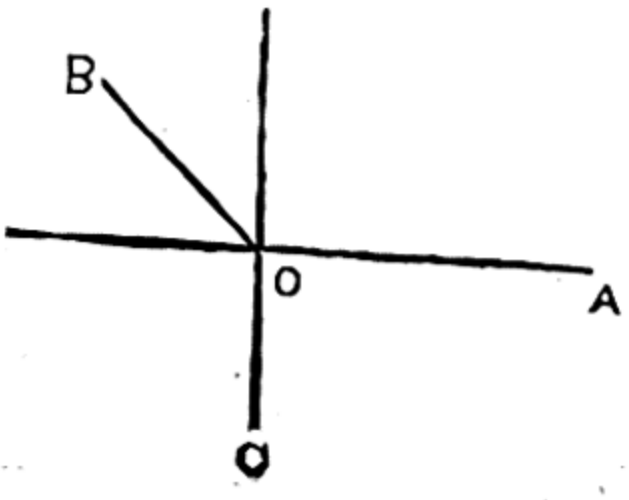
6. The method of '*vectors*' may be employed in a few typical cases, as the following examples will show :—

Ex. (1) : To solve $2z^2 + 2z + 1 = 0$ ... ... (i)

Let OB represent the vector z, then z^2 is represented by OC, where $\hat{AOB} = \hat{BOC}$, and $OA.OC = OB^2$; so that the equation is written

$$\overline{OA} + 2\overline{OB} + 2\overline{OC} = 0. \quad \ldots \quad \ldots \quad \ldots \quad \text{(ii)}$$

Fig. 4.



Now, since OB bisects the angle AOC, equation (ii) requires that 2OC should be eqnal to OA.

Putting OB$=r$, we have

$$2r^2=1, \text{ or } r^2=\tfrac{1}{2}.$$

$$\therefore \quad r=\frac{1}{\sqrt{2}}.$$

Also $2r=-2\cos \text{AOB}$, from (ii.)

$$\therefore \quad \cos \text{AOB}=-\frac{1}{\sqrt{2}}$$

$$\therefore \quad \text{AOB}=135^\circ.$$

Hence, the valne of z corresponding to OB is

$$r(\cos 135^\circ+i\sin 135^\circ)=-\frac{1}{2}(1-i).$$

Ex. (2) $z^3-9z-12=0$.

Here z is easily written in the form $[\sqrt[3]{3}+\sqrt[3]{9}]$ and the several valoes of z are therefore

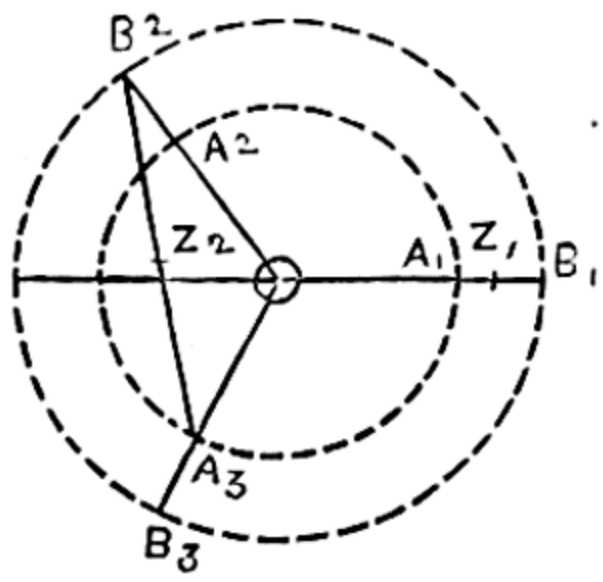
$$a+b,\ a\alpha+b\alpha^2,\ a\alpha^2+b\alpha,$$

where $a=\sqrt[3]{3}$, $b=\sqrt[3]{9}$, α = cube root of unity.

Hence, we obtain the following construction for the roots of the cubic.

Describe concentric circles of radii a, b; inscribe equilateral triangles $A_1A_2A_3$ $B_1B_2B_3$ in them as in the figure.

Fig. 5.



Then the roots are represented by

$$2\,OZ_1,\ 2\,OZ_2,\ 2\,OZ_3$$

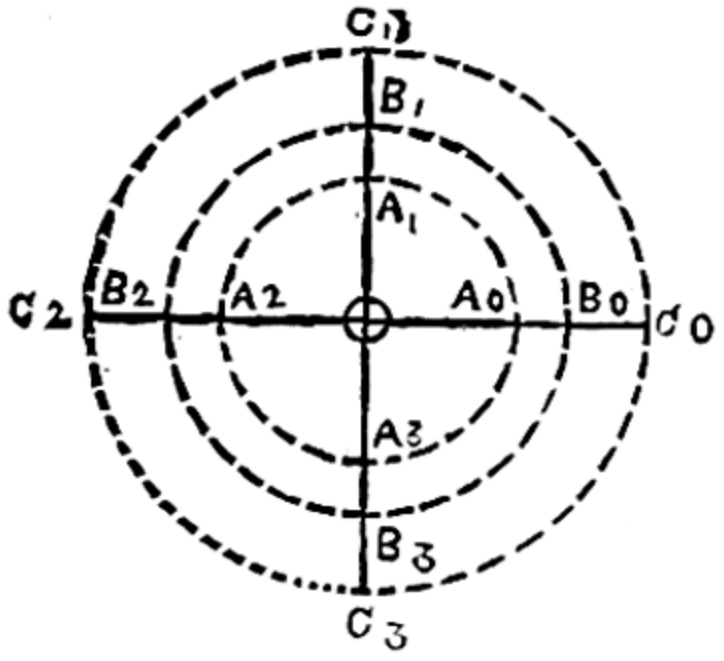
where Z_1 Z_2 Z_3 are the mid-points of A_1B_1, A_2B_3, A_3B_2

Ex. 3. $z^4-18z^2-48z-39=0$.

In this cas e $z=a\alpha+b\alpha^2+c\alpha^3$, where $a=3^{\frac{1}{4}}$, $b=3^{\frac{1}{2}}$, $c=3^{\frac{3}{4}}$, and $\alpha^4=1$. construction for the several values of z corresponding to the four valnes of α is as follows;

Draw concentric circles of radii a, b, c; inscribe squares in them as shewn in the figure. Then the four roots are represented by $3\,OZ_1$, $3\,OZ_2$, $3\,OZ_3$, $3\,OZ_4$, where $Z_1\ Z_2\ Z_3\ Z_4$ are the mean centres of $A_1B_2C_3$, $A_2B_0C_2$, $A_3B_2C_1$, $A_0B_0C_0$

Fig. 6



7. The *vector* method can be successfully employed in the case of a *circulant* equation of any degree whatever.

For, we know that the roots of such an equation of the n^{th} degree can be written

$$a\alpha + b\alpha^2 + \ldots \qquad l\alpha^{n-1}$$

where $\alpha^n = 1$. [Burnside and Panton: *Theory of Equations*, Vol. II, p. 62.]

To represent the roots graphically draw concentric circles of radii $a,b,c,\ldots l$ and place in them regular n—sided polygons

$$A_1A_2\ldots A_n,\ B_1B_2\ldots B_n,\ldots\ldots L_1L_2\ldots L_n.$$

Then the roots of the circulant equation are given by

$$nOZ_1,\ nOZ_2,\ldots\ldots nOZ_n$$

$Z_1, Z_2\ldots\ldots Z_n$ being the mean centres of $A_1B_2\ldots L_n, A_2B_4C_6\ldots, A_3B_6C_9\ldots$, the suffixes having a period n.

8. The *general* cubic and biquadratic equations can also be solved by the above method.

The circulant equation of the *third* degree is

$$\begin{vmatrix} -x & a & b \\ b & -x & a \\ a & b & -x \end{vmatrix} = 0,$$

which is the same as

$$x^3 - 3abx - a^3 - b^3 = 0;$$

comparing with $x^3 + qx + r = 0$, we have

$$q = -3ab,\ r = -(a^3 + b^3).$$

Thus a^3, b^3 are the roots of

$$t^2+rt-q^3/27=0;$$

and the graphical representation of x is as in § 6, Ex. 2.

Again, the circulant equation of the *fourth* degree is

$$\begin{vmatrix} -x & a & b & c \\ c & -x & a & b \\ b & c & -x & a \\ a & b & c & -x \end{vmatrix} = 0,$$

which reduces to

$$x^4-2x^2(b^2+2ac)-4bx(a^2+c^2)+b^2(b^2-4ac)-(a^2-c^2)^2=0.$$

Comparing with $x^4+qx^2+rx+s=0$, we have

$$q=-2(b^2+2ac),\ r=-4b(a^2+c^2),$$
$$s=b^2(b^2-4ac)-(a^2-c^2)^2.$$

Eliminating a and c, the cubic equation for b^2 is

$$t^3+\tfrac{1}{2}qt^2+(\tfrac{1}{16}q^2-\tfrac{1}{4}s)t-\tfrac{1}{64}r^2=0,$$

which is identical with the '*auxiliary cubic*' of the biquadratic.

The difference between Euler's method and this method, however, consists in the expression for the roots of the quartic in terms of those of the cubic. According to Euler's method the roots of the quartic are

$$\sqrt{t_1}+\sqrt{t_2}+\sqrt{t_3}$$
$$\sqrt{t_1}-\sqrt{t_2}-\sqrt{t_3}$$
$$-\sqrt{t_1}+\sqrt{t_2}-\sqrt{t_3}$$
$$-\sqrt{t_1}-\sqrt{t_2}+\sqrt{t_3};$$

whereas our method gives them in the form

$$-b\pm(a-c)i,\ b\pm(a+c),$$

b, a, c standing for

$$-\sqrt{t_1},\ \tfrac{1}{2}\sqrt{t_3}(1-i)-\tfrac{1}{2}\sqrt{t_2}(1+i),\ \tfrac{1}{2}\sqrt{t_3}(1+i)-\tfrac{1}{2}\sqrt{t_2}(1-i)$$

respectively.

Section (II).

9 We shall consider in this Section *approximations* to complex roots of an algebraic equation by means of Cauchy's theorem, *viz.*:

$$f(z)=f(a)+(z-a)f'(a)+\frac{(z-)}{2!}f''(a)+\ldots$$

This series leads to the following approximations, provided there is a root in the neighbourhood of a point a.

The *first* approximation is $(a+h_1)$, such that

$$f(a)+h_1f'(a)=0.$$
$$\therefore\ h_1=-f(a)/f'(a). \qquad \text{(i)}$$

The *second* approximation is $(a+h_1+h_2)$, such that

$$f(a+h_1)+h_2 f'(a+h_1)=0,$$

i.e. $$f(a)+h_1 f'(a)+\frac{h_1^2}{2!}f''(a)+h_2 f'(a)=0.$$

$$\therefore \qquad h_2=-\tfrac{1}{2}[f(a)]^2 . f''(a)/[f'(a)]^3. \qquad \text{(ii)}$$

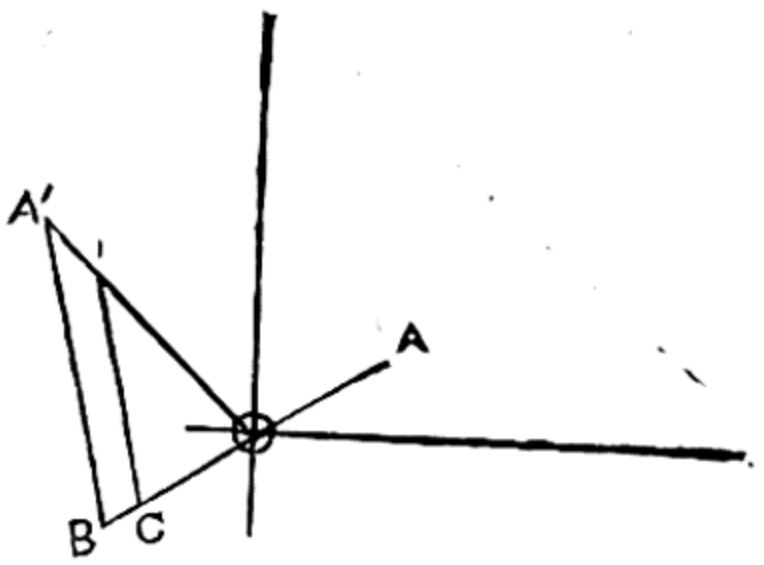
And so on.

These approximations are *graphically* equivalent to the following construction:

Corresponding to a in the z—plane, let

A be the point $f(a)$ in the w—plane,

A′..... $f'(a)$ in the same,



then $$h_1=-f(a)/f'(a)$$

$$=\overline{OB}/\overline{OA}'. \quad [\text{Make } OA^1 . O1=OB^2=OA^2.$$

In other words, h_1 is the vector equal to OC in the figure whose vectorial angle is equal to A′OB.

The *first* approximation to the root is thus

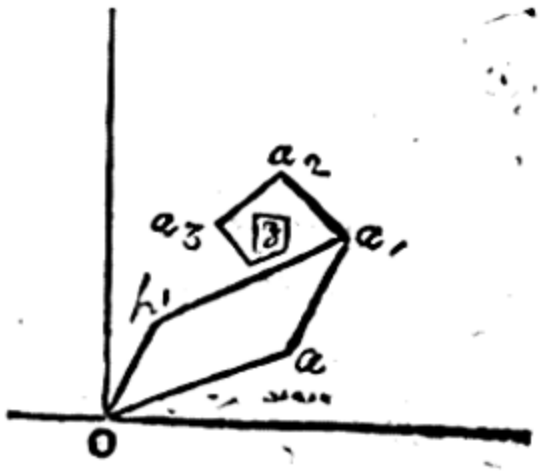
$$a+h_1=a_1 \text{ say.}$$

Similarly, the *second* approximation is

$$a+h_1+h_2=a_2, \text{ say.}$$

And so on, for further approximations.

Fig. 8.



The actual complex root z of the proposed equation is therefore graphically determined as the point of convergence of the polygon $o\ a\ a_1\ a_2 \ldots$

9·1. The assumption that

$$a_1 = a + h_1$$

is a better approximation than a depends upon the following results:

(i) $$\frac{f'(z)}{f(z)} \equiv \sum \left(\frac{1}{z-\alpha}\right)$$

where α is a complex root or real root of $f(z)$. Now, when a is an approximate complex root, the corresponding partial fraction on the right side will have its modulus very great when $z=a$.

Hence $\left|\frac{f'(a)}{f(a)}\right|$ is large when a is an approximate complex root.

that is, $|h_1| = -\left|\frac{f(a)}{f'(a)}\right|$ is small.

(ii) $$f(a_1) = f(a) + h_1 f'(a) + \frac{h_1^2}{2!} f''(a) +$$

$$= \frac{h_1^2}{2} f''(a), \text{ since } h_1 = -f(a)/f'(a).$$

$$\therefore \quad \frac{f(a_1)}{f(a)} = \frac{h_1^2}{2} \frac{f''(a)}{f(a)}.$$

And in general $\left|\frac{f''(a)}{f(a)}\right|$ is small for a similar reason as in (i).

Hence $\left|\frac{f(a_1)}{f(a)}\right|$ is small.

Thus $f(a_1)$ is a better approximation than $f(a)$ under the circumstances.

9·2. The degree of approximation in taking $a_1 = a + h_1$ instead of a may be investigated as in the usual methods for real roots.

10. The approximations found in the last article may also be *directly* obtained by separating the real and imaginary parts at each stage of the approximations.

Thus, putting $$f(a) = \alpha + i\beta,$$
$$f(a) = \alpha' + i\beta', \text{ \&c.,}$$

we have

$$w = u + iv = f(x+iy)$$
$$= \alpha + i\beta + (x+iy-\alpha-i\beta)(\alpha'+i\beta') + \dots$$

Equating real and imaginary parts:

$$u = \alpha + [(x-\alpha)\alpha' - (y-\beta)\beta'] + \dots$$
$$= \alpha + \alpha_1 + \alpha_2 + \dots, \text{ say;}$$
$$v = \beta + [\beta'(x-\alpha) + \alpha'(y-\beta)] + \dots$$
$$= \beta + \beta_1 + \beta_2 + \dots, \text{ say.}$$

The *first* approximation is thus the intersection of the straight lines

$$\alpha+\alpha_1=0,\ \beta+\beta_1=0,$$

which are obviously rectangular.

It may be observed that *all* the successive approximations are *orthogonal*, being *conjugate* functions.

11. The approximations here investigated are capable of interpretation by means of 'the theory of polars' of a binary form.

For, corresponding to any point a, the $(n-1)th$ polars with respect to u and v are easily seen to be identical with the *first* approximations in § 10; and therefore, the first approximation is geometrically interpreted as the point of intersection of these polar lines, which are at right angles. Similarly, for further approximations.

12. A general method of dealing with the intersections of two plane algebraic curves suggests itself from the preceding.

Take *any* point in the neighbourhood of a point of intersection and construct the *polar lines* of the point with respect to each curve. Then the point of intersection of these polar lines will be a *first approximation*; similarly we can construct further approximations.

Section (III).

13. In this last Section we shall apply the method of approximations to investigate the complex roots of a *Transcendental Equation*.

Consider the equation

$$z=\cos z,$$

which readily breaks up into the two equations

$$P\equiv x-\cos x\ \cosh y=0,$$
$$Q\equiv y+\sin x\ \sinh y=0.$$

The P and Q curves are traced by writing them in the forms

$$\cosh y=x\ \sec x,$$
$$\sin x=-y\ \operatorname{cosech} y;$$

and using the *lambda* table (*vide*: A. Lodge's *Diff. Calc.*, p. 82), along with the ordinary mathematical tables.

A tracing of portions of P and Q prepared with reference to the subjoined tabular values, follows:

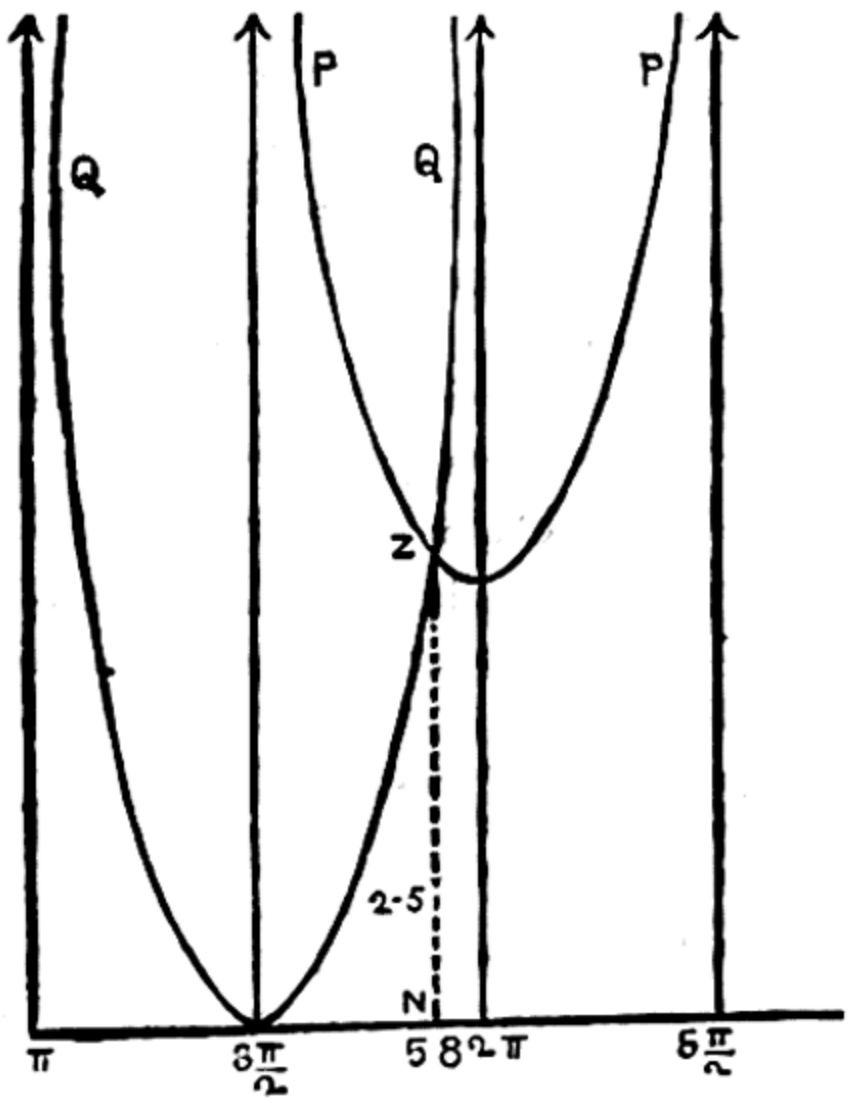
Table (1): $\cosh y=\sec\theta=x.\ \sec x.$

x	$3\pi/2$	6·11	2π	$5\pi/2$	For values of x between $3\pi/2$ & $5\pi/2$.
$\cos\theta$	0	·16	·156	0	
θ	90°	86°40′	81°	90°	
y	∞	2·5 (min.)	2·54	∞	

Table (2) : $\sin x = -y \operatorname{cosech} y$; $\sinh y = \tan \theta$

y	0	1	2	3	∞	For values of x between $\frac{3\pi}{2}$ and 2π.
θ	0°	49°36′	74°40′	84°17′	90°	
$\sin x$	−1	−·851	−·558	−·3003	0	
x	$3\pi/2 = 270°$	301°40′ 238°20′	326°3′ 213°57′	342°31′ 197°29′	360° 180°	

Fig. 9.



From the diagram it is obvious that the P and Q curves cross in the neighbourhood of the point

$$[x = 5{\cdot}8 \; ; \; y = 2{\cdot}5].$$

To determine the *first* approximation, we take the equations

$$P' + h\frac{dP'}{d\alpha} + k\frac{dP'}{d\beta} = 0,$$

$$Q' + h\frac{dQ'}{d\alpha} + k\frac{dQ'}{d\beta} = 0 ;$$

where P′, Q′ stand for the values of P and Q, when $x = 5{\cdot}8 = \alpha$, say ; $y = 2{\cdot}5 = \beta$, say.

Writing $$\sin \alpha = \sin 5{\cdot}8 = \sin 332^\circ\ 19' = -\ {\cdot}4645$$
$$\cos \alpha = \cos 332^\circ\ 19' = {\cdot}8855$$
$$\operatorname{sech} \beta = \operatorname{sech} 2{\cdot}5 = \cos 80^\circ\ 40' = {\cdot}1622$$
$$\tanh \beta = \sin 80^\circ\ 40' = {\cdot}9868$$

from the tables and solving the two equations for h and k, we get

$$h = {\cdot}074,\ k = {\cdot}039.$$

Hence, a first approximation to the complex root in the neighbourhood of (α, β) is

$$x = 5{\cdot}874;\ y = 2{\cdot}539;$$

or $$z = 5{\cdot}874 + 2{\cdot}539\ i.$$

14. We might have proceeded more *directly* thus—

$$f(z) \equiv z - \cos z = (a - \cos a) + (z - a) f'(a), \text{ approximately,}$$

where $a = \alpha + i\beta$; so that

$$z - a = -(a - \cos a)/f'\ a) = -(a - \cos a)/(1 + \sin a)$$
$$\therefore\ z = a - (a - \cos a)/(1 + \sin a) = (a \sin a + \cos a)/(1 + \sin a)$$
$$= [(\alpha + i\beta) \sin (\alpha + i\beta) + \cos (\alpha + i\beta)]/[1 + \sin(\alpha + i\beta)]$$
$$= \alpha_1 + i\beta_1, \text{ say.}$$

Calculating α_1, β_1, we get for the first approximation

$$5{\cdot}874 + 2{\cdot}537i$$

Other complex roots may be obtained similarly by drawing the remaining branches of the P and Q curves and noting their intersections.

Postscript.

15. Two simultaneous equations involving *two variables* are usually represented by plane graphs and their real solutions obtained as the intersections of the graphs. When there are no *real* intersections, however, the corresponding *complex* solutions of the equations may be visualized as follows :—

(i) *First Method.*—Consider z, z' as the unknown complex quantities satisfying the equations

$$f(z, z') = 0,\ \phi(z, z') = 0.$$

Write $z = x + iy$, $z' = x' + iy'$ in the above and separate the real and imaginary parts, so that, we have *four* equations in the *four* unknowns (x, y, x', y').

Now, these four unknowns may be regarded as determining a *real point* in space of four dimensions, and the resulting equations in them as relating to four hyper-surfaces. The common points of these hyper-

surfaces would consequently represent the complex solution of the proposed equations.

The difficulty of conceiving four dimensional space detracts from the usefulness of this method of representation.

(ii) *Second Method.*—In this method z, z' may be associated with a line in space in as much as *four co-ordinates* are necessary to determine a line completely. Accordingly the two proposed simultaneous equations on being split up into real and imaginary parts specify four *complexes* in space, and the complex solutions of the equations are geometrically represented by the lines common to the four complexes. The intuitive difficulties of the previous method exist in this case also, in so far as line-geometry and hyper-geometry are intimately connected.

(iii) *Third Method.*—This is only a relic of the line-representation, being a representation of the two variables z and z' by *a point pair* in the same or in different planes. A few simple properties of the two plane representation are discussed in Forsyth's *Theory of Functions of Two Complex Variables*, § § 15—20.

(iv) *Fourth Method.*—Lastly, we may eliminate one of the variables z', between the proposed equations and treat the resulting equation by the methods applicable to functions of a single variable.

16. In simple cases, however, the *vector method* will be found quite suitable, as the following examples will prove :—

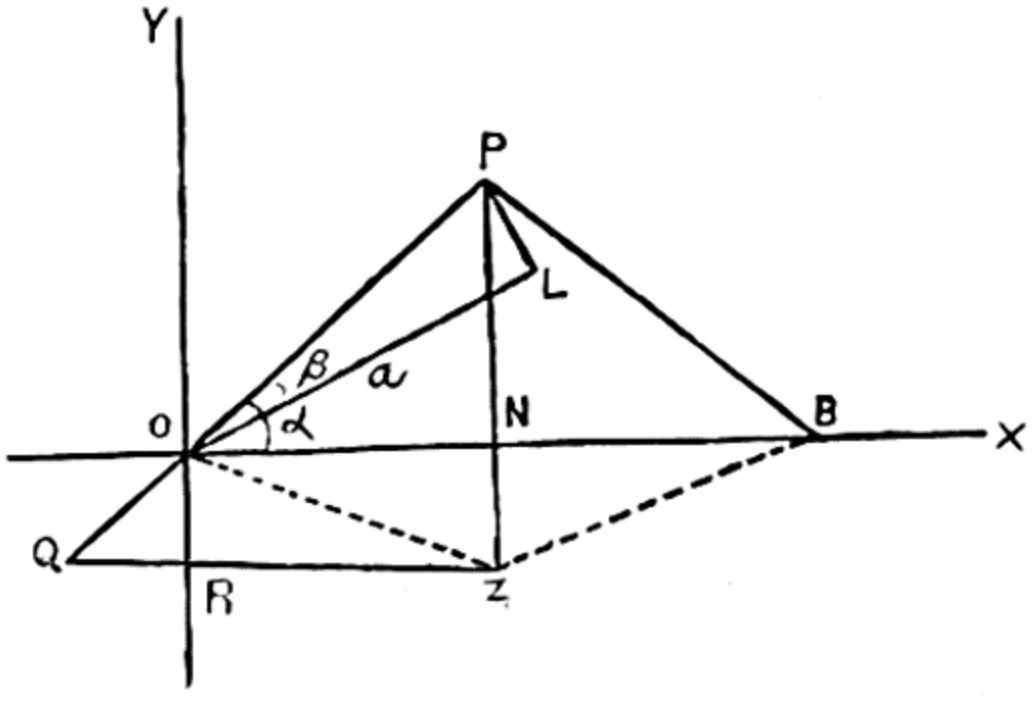
Example 1.—Solve $z^2+z'^2=a^2$, $z+z'=b$; where $b^2>2a^2$.

Let $z=a\cos\theta$, $z'=a\sin\theta$, where θ is a complex quantity. Then we may write

$$z=a(\cos\alpha\sec\beta-i\sin\alpha\tan\beta),$$
$$z'=a(\sin\alpha\sec\beta+i\cos\alpha\tan\beta),$$

α, β being real quantities.

Fig. 10.



Hence the following construction for the vectors z, z'.—
In the figure $\overline{OL}=a$, $OB=b$, $OQ=PL$, &c.; and we have

$$ON=OL \sec \cos \alpha = a \sec \beta \cos \alpha,$$
$$NZ=OQ \sin \alpha = PL \sin \alpha = a \tan \beta \sin \alpha.$$
$$\therefore \quad \overline{OZ}=\overline{ON}+\overline{NZ}=a(\cos \alpha \sec \beta - i (\tan \beta \sin \alpha)$$
$$=z.$$

Again $NP=a \sec \beta \sin \alpha.$
$QR=a \tan \beta \cos \alpha$;

$$\therefore \quad iz'=\overline{NP}+\overline{RQ}.$$

But $z'=b-z=\overline{ZB}=\overline{ZN}+\overline{NB}$

$\therefore$ $NB=NP$, and $ZN=RQ$.

In other words, $\alpha=\pi/4$, $ON=\frac{1}{2}b$.

The vector solutions are thus OZ, ZB, corresponding to the complex quantities

$$[\tfrac{1}{2} b \pm i \sqrt{(\tfrac{1}{2} b^2-a^2)}]$$

Example 2. Solve $z^2/a^2+z'^2/b^2=1$, $z+z'=c$; where $c^2 > a^2+b^2$.

Here, we can write

$$z=a (\cos \alpha \sec \beta - i \sin \alpha \tan \beta)$$
$$z'=b (\sin \alpha \sec \beta + i \cos \alpha \tan \beta).$$

Proceeding, as before,

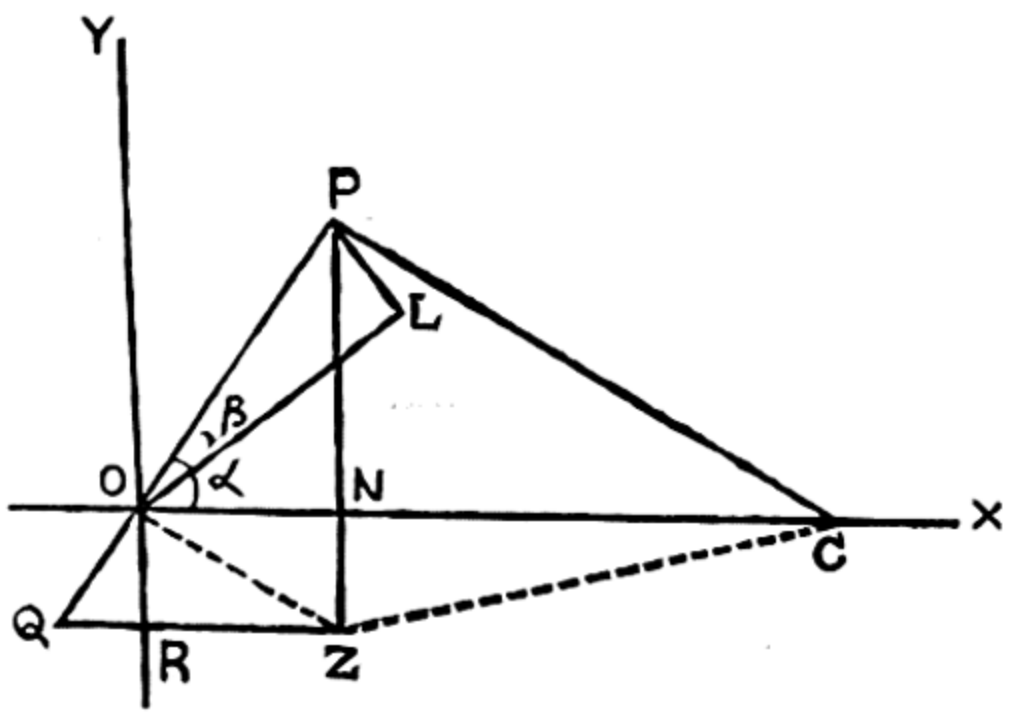


Fig. 11.

$$z=\overline{OZ},\ z'=\overline{ZC}=\overline{ZN}+\overline{NC}.$$

Also $$i z'=\frac{b}{a}\left(\overline{NP}+\overline{RQ}\right).$$

Hence $$NC=\frac{b}{a} NP,\ ZN=\frac{b}{a} RQ.$$

$$\therefore OPC=90°, \text{ ian } \alpha=\frac{b}{a},\ a=c \cos \alpha \cos \beta ; \text{ \&c.}$$

The vector solutions are thus OZ, ZC, corresponding to the complex quantities

$$\frac{a^2 c}{(a^2+b^2)} \pm i \frac{ab \sqrt{(c^2-a^2-b^2)}}{(a^2+b^2)}$$

The Function of Mathematics in Scientific Research.

By Professor G. A. Miller.

(*Concluded from p. 225 of Vol. IX.*)

This common ground of investigators may serve to explain the fact that many of the most influential research organizations, like the National Academy of Sciences in our own country, embrace all the sciences. In recent decades there has been a tendency to organize research separately in the various subjects in the form of national societies named after these subjects. In fact, there are those who think that the latter have assumed such a preponderant sphere of influence as to threaten the very life of the former as serious factors in research. On the other hand, the maintenance of a common scientific life seems to be of the highest importance in view of desirable interactions and special emphasis on what is most fundamental.

The history of mathematics has taught us that some subjects which were apparently far apart and which were long developed separately were later seen to have most important common elements. The discovery of those common elements and their development has led to marked advances in the separate fields themselves. By way of illustration I need only refer to the fields of Algebra and geometry so happily welded through the work of Descartes, Fermat and many others. In modern times the theory of groups and invariants has exhibited many important connections between subjects which had been supposed to be widely separated. The same tendency has, of course, manifested itself in other sciences and may be assumed to become more dominant as knowledge advances.

A pertinent difference between the mathematical investigators and investigators in other sciences is that the former are compelled to stay with their problems until a solution is reached which can be proved to be in accord with deductions from certain definite assumptions, while the latter enjoy much greater freedom in regard to the stage to which they may pursue a problem before announcing results. Hence these may hope for success in dealing with much more difficult questions than the mathematician could reasonably hope to solve at the present time. The limitations thus imposed upon the mathematician are compensated, at least in part, by the finality of his results as regards questions of rigor. Mathematical results *can* never be disproved, other accepted results *have* never been disproved. With respect to simplicity and style, the mathematical developments are seldom final, and in many cases, they appear to admit endless variations.

As instances of final mathematical results may be cited the useful tables which when once computed serve all succeeding generations. Such finality may be said to be a goal of all scientific endeavor, since the results enrich countless ages by increasing their capacity for accomplishments. In fact, such tables may be regarded as typical illustrations of the mathematical contributions to the advancement of knowledge even if they constitute a very minor portion of these contributions. The fact that mathematical results have increased the capacity of the world for doing things may be emphasized by noting, in particular, that in recent years prime numbers have been found which could not have been proved to be prime by the method employed by Eratosthenes, if the entire human race had been working in an organized manner on this single problem since the days of the ancient Greeks.

The present seems to be an especially appropriate time to consider the interrelations of scientific research in view of the rapidly growing public appreciation of the value of such research. Several decades of comparative peace immediately preceding the present great and deplorable conflict were unusually rich in great scientific triumphs. As well-known instances we may cite wireless telegraphy and the construction of the great Panama Canal, which became possible by our advanced knowledge in regard to sanitation. The world-wide health activities under the auspices of the Rockefeller Foundation and the activities of our agricultural colleges in directing attention to advantages resulting from scientific methods of farming are strong forces working towards a popular appreciation of science. Since the great European war began it has become evident through the new elements introduced by the submarines and other scientific devices that the very existence of a great nation may depend upon the scientific attainment of its people, and hence the question of scientific research has taken a prominent place among those of national policy. It is perhaps significant that our National Academy was founded in the midst of the Civil War.

Scientific research is as old as civilization and has often been protected by kings in a patronizing manner, but it is a new experience in the history of the world to see kings turn to scientific research for protection. For centuries governments have recognized the value of science and have provided with growing liberality for her development, but now they are calling to her to save them from destruction. They have noticed that in spite of many excellencies in other directions the ignorance of causes may entail their destruction as separate nations. This

new attitnde towards our field of work may at first tend to gratify us, but a second thought reveals the fact that it is franght with grave dangers. Kings in government and finance are interested in the dead results of science instead of in the great living and growing organism itself, whose growth seems to have just begun and whose development has always been more keenly inspired by love of truth than by hope of gain.

Is there not a danger that the sudden recognition of the great political importance of certain types of research will have somewhat the same effect on science as the discovery of gold in California and in Australia about the middle of the preceding century had on the development of the regions concerned? People flocked from one mining camp to the other and often neglected duties which are essential for the harmonious development of the resources of a country. Hence there seems to be a special need at present to urge our colleagues to remain at their posts of duty, notwithstanding glowing reports of chances to amass scientific fortunes quickly in certain newly discovered gold fields. The get-rich-quickly schemes in science should be scrutinized as carefully as similar schemes relating to the accumulation of money.

The remaining at one's post of duty in scientific research does not imply a lack of support in the solution of pressing problems or a lack of vacation trips and acquaintance with other fields of work. In fact, such support and acquaintance are highly desirable. It is, however, a question whether the nomadic scientific life, which seems to have become fashionable during the last few decades, at least in mathematics, is the one which will in the long run bring the best results. Science is not primarily a grazing country. Large tracts are suited for agriculture and mining. What is new is not necessarily good and what is good is not necessarily new, and prophesies in regard to the great importance of certain new developments have not always been fulfilled. On the other hand, it should be remembered that reasonable hope and optimism are essential for progress, and that we need prospectors as well as miners in the scientific world.

It should be noted that the miner needs some of the qualifications of the prospector since he is apt to meet with new situations and needs to take advantage of the available by-products. In fact, while he is mining for gold he may strike deposits of copper which are richer than the gold deposits which he was primarily seeking. Some of the richest mathematical discoveries were made while the investigator was looking primarily for other results, and even problems which have not been

solved at all up to the present have been the source of very useful developments. I understand that similar conditions hold in other fields of scientific effort and these facts point to the great importance of freedom on the part of the investigator, and, incidentally to the danger of too much organization in scientific research.

As a very recent instance of an unexpected mathematical by-product, I may be pardoned for referring to a somewhat trivial case which has, however, the important property that it can be understood by all. It is well known that the theory of substitution groups was developed for the purpose of clarifying the theory of algebraic equations and not for the purpose of adding to the enjoyment of parties engaged in playing games of cards. In fact, the study of such an advanced mathematical theory as that of substitution groups might appear to involve concepts, which are at the opposite pole from those entering the minds of people seeking recreation at card tournaments.

Notwithstanding this apparent wide separation, I was pleased to be able to say recently to a friend, who desired to have each one of a large party play once and only once with each of the others during a series of successive games, that an arraugement of the players meeting this condition could be determined directly by means of substitutions of certain transitive groups. This should perhaps have been expected, since a transitive substitution group is an ideal republic treating all its etters in exactly the same way. On the contrary, an operation group may have elements enjoying special privileges and hence it has more extensive contact in the actual world of thought.

A little study of the stated problem revealed the interesting fact that when the number of tables is any power of 2 the substitutions of a well-known type of substitution groups and its group of isomorphisms exhibit directly how the players can be arranged so that each one will play once and only once with, and twice and only twice against, each of the others in a certain series of games. To make myself perfectly clear, I may say that if 8 tables, or 32 players, are involved, one can write directly by means of a certain regular substitution group of order 32 a set of possible arrangements so that in 31 successive games each one of these 32 players would play once and only once with each of the others and twice and only twice against each of them. This was, however, not the first solution of the general problem in question. In fact, about twenty years ago Professor E. H. Moore published a different solution of it in Volume 18 of the *American Journal of Mathematics* under the title "Tactical Memoranda,"

I have referred to this matter here mainly for the purpose of emphasizing the fact that intellectual penetration is often attended by the most unexpected by-products, but I should also be pleased to have people know that certain kinds of recreation can easily be enriched by making use of results which the mathematician developed for a totally different purpose. Science should and does enrich both work and play. More than a thousand years ago the Hindu astronomer Brahmagupta said:

As the sun obscures the stars, so does the proficient eclipse the glory of other astronomers in an assembly of people by the recital of algebraic problems, and still more by their solutions.*

The playful question, Where do the finger nails find so much dark dirt to put under them? may serve to arouse a thoughtful attitude on the part of the boy who has been taught to keep his hands clean. In fact, our play and recreation are perhaps as fundamentally affected by questions of science as our serious work and the victrolas and moving pictures should have a marked influence on the popular attitude towards science in view of the fact that they reach so many people. If it is true that the greatest service which science is rendering the human race is the reduction of superstition, it is clear that the efficiency of science depends largely upon its popularity.

The hypothesis that space and the operations of nature are discontinuous clearly excludes the hypothesis that they are continuous, but it is interesting to note that the mathematics relating to the discontinuous does not exclude that relating to the continuous. On the contrary, there are the most helpful interrelations between these two types of mathematics. Such a subject as number theory, relating decidedly to discrete quantities, has been greatly extended by analytic methods relating to continuous quantities, and, on the other hand, processes relating to the study of continuous functions are largely based upon those relating to the discontinuous.

This may perhaps tend to show that even if our hypotheses in regard to the continuity of space and the operations of nature have to be largely modified, as seems now probable, the mathematical methods of attack may require less modification than might at first appear to be necessary. The language which mathematics has provided for science includes not only concepts relating to the continuous and the discontinuous, but fortunately it also shows relations between these concepts and these relations become more pronounced with its development.

* H. T. Colebrooke, "Algebra with Arithmetic and Mensuration from the Sanskrit," by Brahmagupta and Bhascara, 1817, p. 379.

In view of the age of this language and its contact with various sciences it may be readily understood why mathematical history occupies a prominent place in the history of science. In fact, the history of science constitutes one of the fields where scientists may find common interests most fully represented, even if the past is too rich in events to be studied completely. It may therefore be appropriate on this occasion to refer to a few recent developments relating to the history of mathematics, especially since the interest in the history of science has increased rapidly during recent decades, as is partly evidenced by the efforts that are now being made to establish an institute of historical scientific research in our land.

One of the most interesting questions relating to the early history of mathematics is the use of positional values of numbers and the closely connected use of a symbol for zero. Until a decade or two ago it was commonly assumed by mathematical historians that the use of zero as a positional number symbol originated in India, and this view has not yet been entirely abandoned, notwithstanding the fact that the Babylonians employed numbers with positional value and a symbol which seems to have fulfilled the main function of our zero several centuries before the Christian era. On the other hand, the first definite evidence of the use of zero among the Hindus falls in the second half of the first millennium of this era.

In view of these facts it is extremely interesting to note the early use of zero, in connection with numbers having positional value, by the Maya, a people inhabiting the Atlantic coast plains of southern Mexico and northern Central America. One of the worthy alumni of your university recently referred to this matter in the columns of SCIENCE in the following words:

Special interest attaches to the occurrence of zero-symbols and the principle of local value among the inhabitants of the flat lands of Central America, at a period as early as the beginning of the Christian era, if not much earlier. It would seem that in this invention, the Maya in Central America possessed priorty over Asiatic people by a margin of five or six centuries.

If further investigation will lead mathematical historians to agree that the zero as a symbol in a numerical notation with positional value was actually first used in America, according to the preserved records, it will effect a very fundamental change as regards interest in the early mathematical attainments of the American aborigines. Unfortunately these early mathematical attainments failed to become the source of extensive further developments on American soil. They exhibit clearly

that central concepts may be discovered independently and they direct attention to the danger in trying to establish one source for a particular concept in historical insvestigation. They also show that the small strip of country marked now by Boston has not always been the intellectual hub of America.

The history of some of the mathematical attainments of the Maya people has recently been made more easily accessible through the publication of "An Introductiou to the Study of the Maya Hieroglyphs," prepared by S. G. Morley and published as Bullettin 57 of the Bureau of American Ethnology, Smithsonian Institution of Washington. On page 92 of this bulletin a dozen different symbols for zero are noted and on page 131 numbers varying from 21 to 12,489,781, and involving the use of zero, are represented in the Maya notation. It is of interest to note that the value of a unit in a higher position is always 20 times the value of a unit in the next lower position, except in the case of the third place, where its value is only 18 times that of the second place.

In historical research and elsewhere,the mathematician seeks cordial cooperation with other scientists, and he regrets that the confusion of tongues, resembling the experiences at the tower of Babel, is making it more and more difficult to understand each other. In the case of scientists this confusion is mainly due to a rapid growth of language in various directions. May we not hope that as many theories which were supposed to be distinct suddenly exhibited profound connections, so also this extensive language will tend towards unity and simplicity as we see more clearly the fundamental underlying principles. Science knows no bounds in method or in subject-matter and the artificial limitations set by man for his own convenience in making a start must break down before the onward march of truth. All science is a unit and all scientific investigaton should be inspired by their common interests.

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[7] F. Cajori. SCIENCE, N. S., Vol. 44 (1916), p. 715.

Astronomical Notes.

Eclipses.

The first eclipse of the year 1918 will occur on June 8, when there will be a total eclipse of the Sun. The eclipse will be visible as a partial eclipse from the north-eastern part of Asia, the north polar regions and the whole of the north American continent. The track of totality runs from a point in the Bahamas Islands nearly to the East coast of China; as it runs right across the centre of the United States the eclipse is likely to be well observed in spite of the war; the remainder of the central track lies in the Pacific Ocean.

There will be a partial eclipse of the Moon on the night of June 23—24, the eclipse is a small one, magnitude 0·135 and is invisible in India.

R. A. S. Gold Medal.

The Gold Medal of the Royal Astronomical Society was awarded this year to Mr. J. Evershed, Director of the Solar Physics Observatory at Kodaikanal.

Astronomical Consequence of a Curvature of Space.

In Mon. Not. R. A. S. 1917 Nov. de Litter investigates the effect on astronomical phenomena of a curvature of space taken in conjunction with Einstein's theory.

It is quite clear that two dimensional beings living on the surface of a sphere of very large radius might conceive themselves to be resident on a plane, and a similar notion may be extended to space in three dimensions, but by extending our observations to a distance comparable with the radius of curvature, we may hope to detect the difference; hence it is to astronomical observations that we must look for any effect of curvature of space.

The new solution of Einstein's equations leads to a form of space with constant positive curvature. Such space may be either

(I) Riemann's spherical space in which all straight lines starting from a point intersect again in the antipodal point, whose distance from the starting point is π. R (R = radius of curvature) along all lines, and this is the greatest possible distance between two points; or

(II) Newcomb's elliptical space, in which any two lines have at most one point in common, and the largest possible distance between two points is $\frac{1}{2}\pi$. R.

In (I) All lines are of length $2\pi R$; in (II) all lines are of length πR

de Litter finds that (on the assumption of a curved space) no star can possibly have a parallax less than a/R where a is the distance between the Earth and Sun.

Secondly it is found that the lines in the spectra of very distant stars and nebulæ will be displaced towards the red, producing a spurious positive radial velocity; a similar result holds in rectangular space on Einstein's theory, but the displacement is considerably augmented in a curved space. In this way it may be possible to account for the very large velocities of spiral nebulæ, which are certainly very distant, should they turn out to be positive on the whole. de Litter makes a number of estimates of the value of R, the data of course is somewhat scanty, since the smallest parallaxes are naturally the most uncertain, and very few absolute parallaxes are known, while the number of spiral nebulæ of which we have reliable determinations of radial velocity is very small (de Litter uses only three). R comes out about 10^{13} Astronomical Units.

If space is curved we should see an image of the sun at the antipodal point, that is the point of the sky opposite to the sun, but opinions appear to differ as to whether this image would be as large and bright as the sun or whether it would appear as an extremely faint star.

R. J. Pocock.

SOLUTIONS

Question 329.

(M. Bhimasena Rao) :—If the pedal circle of P with respect to a triangle ABC touches the nine-points-circle of ABC, show that the sum of the angles PAB, PBC, PCA is constant.

Additional Solution by the Proposer.

The following simple result concerning the rectangular hyperbola which may be proved easily by the anharmonic property of conics is here assumed :—

ABCPQ is a rect hyp, perpendiculars QD, QE, QF are dropped on the sides of ABC intersecting PA, PB, PC in A′, B′, C′.

Then $QD.\ QA' = QE.\ QB' = QF.\ QC'$ ... ... (1)

This is the converse of the theorem. 'If two triangles ABC and A′B′C′ are conjugate with respect to a circle, they are in perspective; if P and Q be the centre of perspective and the centre of the circle respectively, the conics ABC PQ and A′B′C′ PQ are rect hyp.

If P and Q be isogonal conjugate points and D′E′F′ the pedal triangle of P

we have $PD'.\ QD = PE'.\ QE = PF'.\ QF$ ... ... (2)

From (1) and (2) we see that when ABC PQ is a rectangular hyp. QA′, QB′ and QC′ are proportional to PD′, PE′ and PF′, and being parallel respectively, the triangles A′B′C′ and D′E′F′ are homothetic. But we know that any inverse triangle of ABC—call it LMN—with respect to P is similar to the pedal triangle of P, the angle of similitude being the complement of $P\hat{A}B + P\hat{B}C + P\hat{C}A$. It follows therefore that A′B′C′ and LMN are similar and being in perspective, one of the two alternatives arises — either they are homothetic, or ABCP is concyclic as also A′B′C′ P. The second alternative being rejected since P is not on the circum circle of ABC, we see that A′B′C′ and LMN are homothetic. Therefore D′E′F′ which is the pedal triangle of P, A′B′C′ and LMN are homothetic taken two by two, and the angle of similitude vanishes.

Hence PAB+PBC+PCA is equal to one right angle.

Corollary. When a triangle ABC and its inverse with respect to a point P are orthol ogic, the pedal circle of P with respect to either triangle touches the nine-point-circle of that triangle.

The above proof fails when P is on the circumcircle of ABC; but in this case the result follows more easily. For properties of the points of intersection of McCay's cubic and the circumcircle of ABC, see Mr. S. Narayanan's paper on 'Three special points' page 85, Vol. I of the Journal,

Question 427.

(S. RAMANUJAN) :—Express

$$(Au^2+Buy+Cy^2)\,(Ap^2+Bpq+Cq^2)$$

in the form $Av^2+Buv+Cv^2$,

and hence shew that if

$$(2x^2+3xy+5y^2)(2p^2+3pq+5q^2)=2u^2+3uv+5v^2,$$

one set of values of u and v, is

$$u=\tfrac{5}{7}\,(x+y)(p+q)-2xp,\ v=2qy-(x+y)(p+q).$$

Solution (1) *by* '*Zero*', (2) *by S. Narayan.*

(1) Let $f(xy)\equiv Ax^2+Bxy+Cy^2\equiv A(x-\alpha y)(x-\beta y)$,

then $f(pq)\equiv Ap^2+Bpq+Cq^2\equiv A(p-\alpha q)(p-\beta q)$,

where α and β are the roots of $A\lambda^2+B\lambda+C=0$

Now
$$f(xy)\cdot f(p,q)\equiv A^2\{(x-\alpha y)(x-\beta y)(p-\alpha q)(p-\beta q)\}$$
$$\equiv A^2\{px+\alpha^2qy-\alpha(py+qx)\}$$
$$\{px+\beta^2qy-\beta(py+qx)\}$$

But $A\alpha^2+B\alpha+C=0$, and $A\beta^2+B\beta+C=0$.

$\therefore$
$$f(x,y)f(p,q)\equiv\{Apx-A\alpha(py+qx)-qy(B\alpha+C)\}\times$$
$$\{Apx-A\beta(py+qx)-qy(B\beta+C)\}$$
$$\equiv A(u-\alpha v)(u-\beta v)$$
$$\equiv Au^2+Buv+Cv^2,.$$

where $Apx-Cqy=u\sqrt{A},\ A(py+qx)+Bqy\equiv v\sqrt{A}.$

A *general* solution can be obtained by the following artifice. Take (a, b) such that

$$f(a, b)=a \text{ square}$$
$$=c^2 \text{ (say)}.$$

Then $f(xy).f(pq).f(ab)=A^3(x-\alpha y)(x-\beta y)(p-\alpha q)(p-\beta q)$
$$(a-\alpha b)(a-\beta b).$$

By virtue of the quadratic $A\alpha^2+B\alpha+C=0$, the product $(x-\alpha y)(p-\alpha q)(a-\alpha b)$ can be reduced to the form $(u-\alpha v)$, where u and v are rational functions of $(x\ y\ p\ q\ a\ b)$

Similarly $(x-\beta y)\,(p-\beta q)\,(a-\beta b)=u-\beta v.$

$\therefore$ $f(x\ y).\ f(p\ q).\ f(a\ b)=A^3(u-\alpha v)\,(u-\beta v)$
$$=A^3(A\,u^2+Buv+Cv^2)$$

i.e., $c^2.\ f(xy).\ f(p\ q)=A^2(A\,u^2+Buv+Cv^2)$

Hence $f(x\ y).\ (f\ p\ q)=A\,u'^2+Bu'\,v'+Cv'^2,$

where $u'=Au/c,\ v'=Av/c.$

The *general* solution of the indeterminate equation

$$f(a, b)=c^2$$

leads to a corresponding *general* solution of the proposed question.

In the particular case of A=2, B=3, C=5, one solution of

$$2a^2+3ab+5b^2=c^2$$

is $a=1$, $b=-1$, $c=2$, and we easily write a set of values of u and v different from those given by the proposer.

(2) $Ax^2+Bxy+Cy^2$ may be easily thrown into one of the three forms

$$(lx+ly)^2+(mx+ny^2,$$
$$(lx+my)^2+(nx+y)^2,\ (lx+my)^2+(x+ny)^2,$$

provided $B^2+4AC<0$.

Similarly $Ap^2+Bpq+Cq^2$ may be expressed.

And since $(a^2+b^2)\ (c^2+d^2)$

can be expressed as the sum of two squares in two ways, it follows that

$$(Ax^2+B\ xy+Cy^2)\ (Ap^2+Bpq+Cq^2)$$

can be expressed as the sum of two perfect squares. Hence expressing

$$Au^2+Buv+Cv^2$$

as the sum of two squares and comparing with the above form, the values of u and v are easily obtained. It is clear that there are 108 ways of doing this. The values of u, v given by the proposer for the particular question constitute one of these ways.

And
$$u=\frac{3}{2}px-\frac{7}{2}y-\frac{1}{8}(x+y)\ (p+q),$$
$$v=\frac{3}{2}qy+\frac{1}{2}px-\frac{11}{8}(x+y)\ (p+q).$$

constitute another solution.

Question 508.

(S. P. Singaravelu Mudaliar):—If s_n stand for the sum of the reciprocals of the first n natural numbers, find the sum of the infinite series.

$$s_1-\frac{1}{2}\left(\tfrac{1}{2}\right)^2 s_2+\frac{1}{3}\left(\frac{1\cdot3}{2\cdot4}\right)^2 s_3-\ldots$$

Remarks by H. Br.

We are required to evaluate

$$S=\sum_{n=0}^{\infty}(-1)^n\ a_n^2\ \frac{s_{n+1}}{n+1}$$

where $a_n=\dfrac{1\cdot3\ldots\ldots(2n-1)}{2\cdot4\ldots\ldots(2n)}$, and $s_{n+1}=1+\frac{1}{2}+\ldots+\frac{1}{n+1}$

It is easy to obtain a variety of definite integrals, (simple, double, and triple) for the value of S, but I have not been able to find a formula giving its numerical value—

(1) As $$-\int_0^1 \log(1-x).\, x^n dx = \frac{s_{n+1}}{n+1},$$

$$S = -\int_0^1 dx \log(1-x).\left(\sum_0^\infty a_n^2 (-x)^n\right).$$

Now $$(1+x\sin^2\theta)^{-\frac{1}{2}} = \sum_0^\infty a_n(-x)^n \sin^{2n}\theta\,;$$

$$\therefore \quad \int_0^{2\pi} \frac{d\theta}{(1+x\sin^2\theta)^{\frac{1}{2}}} = \frac{\pi}{2}\sum_0^\infty a_n^2 (-x)^n,$$

and $$S = -\frac{2}{\pi}\int_0^{\frac{\pi}{2}} d\theta \int_0^1 \frac{\log(1-x).}{(1+x\sin^2\theta)^{\frac{1}{2}}} dx,$$

The integration with respect to x can be easily performed, but the result does not throw any fresh light on the subject.

(2) Again we may take $a_n^2 = \frac{4}{\pi^2}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \sin^{2n}\theta \sec^{2n}\phi \; d\theta \; d\phi$, and we get

$$S = -\frac{4}{\pi^2}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^1 \frac{\log(1-x).\, dx\, d\theta\, d\phi}{1+x\sin^2\theta\cos^2\phi}.$$

(3) We have $$a_{n+1} = a_n \cdot \frac{2n+1}{2n+2} = a_n\left[1-\frac{1}{2(n+1)}\right]$$

$$\therefore \frac{a_n^2}{n+1} = 2(a_n^2 - a_n a_{n+1})$$

$$= \frac{8}{\pi^2}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \sin^{2n}\theta.\, \sin^{2n}\phi.\, \cos^2\phi\, d\phi$$

and $(-1)^n s_{n+1}$ is the coefficient of

$$+t \text{ in } \frac{(t-1)(t-2)(t-3)\ldots(t-n+1)}{(n+1)!}$$

$$\therefore \quad S = \text{coefficient of } t \text{ in } \frac{8}{\pi^2}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \frac{d\theta\, d\phi .\cos^2\phi}{\sin^2\theta \sin^2\phi} \sum_{n=0}^{\infty}(t-1)(t-2)\ldots(t-n-1) \frac{}{(n+1)!}$$

$$\sin^{2n+2}\theta \sin^{2n+2}\phi$$

$$= \text{coefficient of } t \text{ in } \frac{8}{\pi^2}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \frac{d\theta\, d\phi .\cos^2\phi}{\sin^2\theta \sin^2\phi}\left[-1+(1+\sin^2\theta\cos^2\phi)^{t-1}\right]$$

$$= \frac{8}{\pi^2}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \frac{d\theta\, d\phi .\cos^2\phi}{\sin^2\theta .\sin^2\phi} \frac{\log(1+\sin^2\theta\sin^2\phi)}{1+\sin^2\theta\sin^2\phi}.$$

(4) Another from of the integral may be obtained by using the fact that

$$\int_0^{2\pi} \frac{d\theta}{\sqrt{1-2r\cos\theta+r2}} = 2\pi\Sigma a_n^2 r^{2n}.$$

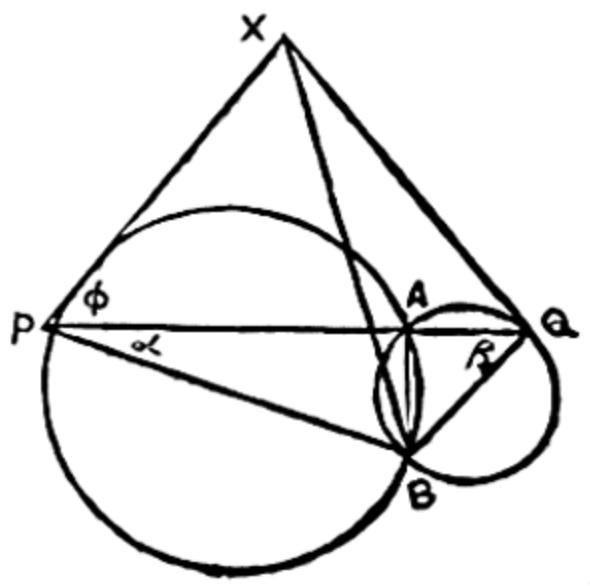
Question 591.

(A. H. KRISHNASWAMI AIYANGAR):—Two circles intersect at A and B; the tangents at the extremities of a double chord through A meet in X. Shew that XY perpendicular to BX envelopes a circle.

Solution by 'Zero'.

Let PAQ be the double chord. then

$$\text{XPA} + \text{XQA} = \text{PBA} + \text{QBA} = \text{PBQ}.$$



$\therefore$ PBQX is cyclic.

Now $BP = AB \sin(\phi+\alpha)/\sin\alpha$

$BQ = AB \sin(\phi+\alpha)/\sin\beta$

Also BP. BQ = BX. BA, since in the cyclic quadrilateral PBQX PBA = ϕ = XBQ.

Hence $BX = AB \sin^2(\phi+\alpha)/(\sin\alpha \sin\beta)$
$= AB \{(1-\cos(2\phi+2\alpha)\}/(2\sin\alpha\sin\beta)$
$\propto \{1-\cos(\pi+\theta-\beta+\alpha)\}$,
where $\theta = ABX = (2\phi-\pi+\alpha+\beta)$.

In other words, the locus of X is the cardioide
$$r = a\{(1+\cos(\theta+\alpha-\beta)\},$$
shewing that the envelope of XY is a circle.

Question 650.

(N. B. Pandya) :—Circumscribe an ellipse about a given triangle so that incentre of ellipse may coincide with the incentre of the triangle.

Solution by C. Bhaskaraiya.

The areal co-ordinates of the in centre are
$$a : b : c.$$

If $Fyz + Gzx + Hxy = 0$ be the required ellipse, the polar *w. r. t.* if of $(a : b : c)$ ought to be identical with the line at infinity, $x+y+z=0$.

$$\therefore \quad cG+bH = Ha+cF = Ga+bF.$$

$$\therefore \quad F : G : H = a(a-b-c) : b(b-c-a) : c(c-a-b)$$

Hence the ellipse is
$$a(b+c-a).yz+b(c+a-b)zx+c(a+b-c)xy=0,$$

To trace the ellipse put $\frac{y}{b}=\frac{z}{c}$, and see where it cuts the bisector of the angle A; and so on.

Question 671.

(K. J. Sanjana, M.A.) :—The numbers from 1 to 2^n-1 being arranged on n cards on which the least numbers are respectively 2^0, 2^1 ... ... 2^{n-1} as in Question 640, prove that the sum of the numbers on the r^{th} card is $2^{2n-2}+2^{n+r-2}-2^{n-2}$

Solution by N. Sankara Aiyar and V. Anantaraman.

The first number on the r^{th} card is 2^{r-1} and all numbers will be found on it which when divided by 2^r leave remainders greater than or equal to 2^{r-1}.

Hence the numbers on that card are

$$\{2^{r-1}+(2^{r-1}+1)+..+(2^{r-1}+2^{r-1}-1)\}+\{(2^r+2^{r-1})+(2^r+2^{r-1}+1)+.\}$$
$$+\{(2\cdot2^r+2^{r-1})+(2\cdot2^r+2^{r-1}+1)+ \quad \dots \quad \dots \quad \} \quad \dots \quad \dots$$
$$+\{(2^n-2^{r-1})+(2^n-2^{r-1}+1)+ \quad \dots \quad +(2^n-1)\} \quad \dots \quad \dots$$
$$=2^{r-1}\times 2^{r-1}(1+3+5+ \quad \dots \quad +2^{n-r+1}-1)\} \dots \quad \dots$$
$$+\frac{2^{n-r+1}}{2}(0+1+\dots \quad \dots \quad +2^{r-1}-1).$$

$$= 2^{2r-2}\cdot\left(\frac{2^{n-r+1}}{2}\right)^2+2^{n-r}\cdot\frac{2^{r-1}}{2}(2^{r-1}-1)$$
$$= 2^{2r-2}\cdot 2^{2n-2r}+2^{n-r}\cdot 2^{r-2}(2^{r-1}-1)$$
$$= 2^{2n-2}+2^{n+r-3}-2^{n-2}.$$

Question 682.

(S. RAMANUJAN) :—Show how to find the cube root of surds of the form $A+\sqrt[3]{B}$; and deduce that

$$\sqrt[3]{(\sqrt[3]{2}-1)}=\sqrt[3]{\frac{1}{9}}-\sqrt[3]{\frac{2}{9}}+\sqrt[3]{\frac{4}{9}}.$$

Solution by 'Zero.'

If α denote a cubic surd any expression in α can be written in the form $(a+b\alpha+c\alpha^2)$ and further

$$(a+b\alpha+c\alpha^2)^3\equiv P+Q\alpha+R\alpha^2 \text{ (say)}.$$

Thns, when either Q or R is zero, we have a binomial cubic surd on the right side.

Hence we can write

$$\sqrt[3]{(A+\sqrt[3]{B})}\equiv r^{\frac{1}{3}}(a+b\alpha+c\alpha^2),$$

and proceed to find the values of a, b, c, α satisfying the following :

$$\left.\begin{aligned} rP&=A\\ Q&=0\\ rR\alpha^2&=B^{\frac{1}{3}}\end{aligned}\right\} \text{ or } \left.\begin{aligned} rP&=A\\ R&=0\\ rQ\alpha&=B^{\frac{1}{3}}\end{aligned}\right.$$

In the particular case stated, $A=-1$, $B=2$;

$$\therefore\quad P\equiv a^3+b^3\alpha^3+c^3\alpha^6+6abc\alpha^2,$$
$$Q\equiv 3a^2b+3ac^2\alpha^3+3b^2c\alpha^3,$$
$$R\equiv 3ab^2+3a^2c+3bc^2\alpha^3,$$

and suitable values for a, b, c, r are seen to be

$$a=-b=c=1,\ r=\frac{1}{9}.$$

More generally if $AQ=BP$, we may put $a=c$ and $\alpha=\sqrt[3]{B}$, so that $R=0$ reduces to

$$a^3+b^3+abB=0$$
$$\therefore\quad b=-m\ a, \text{ if } B=\left(m+\frac{1}{m}\right);$$

and we may, without loss of generality, write $a=c=1$, $b=-m$.

The problem is thus solved, when B is of the form $\left(m+\frac{1}{m}\right)$, and $3A\{B(1-m)-m\}=B\{B^2-B(m^3+6m)+1\}$

Question 725.

(K. B. Madhava) :—Shew that

$$\int^{\infty} \frac{x^2 dx}{1+x^9 \sin^2 x} \text{ converges,}$$

but that

$$\int^{\infty} \frac{x^3 dx}{1+x^8 \sin^2 x} \text{ diverges.}$$

Solution by S. R. Ranganathan.

The condition for the convergence of

$$\int^{\infty} \frac{x\beta}{1+x\alpha \sin^2 x}$$

are fully discussed in Bromwich: *Infinite Series*, App. III. Art. 166. It is proved there that the integral converges or diverges with the series

$$\sum\left(n^{\beta}-\frac{\alpha}{2}\right),$$

i.e., according as $\alpha>2(\beta+1)$ or $\alpha\leq 2(\beta+1)$.

Now taking $\beta=3$, we get that

$$\int^{\infty} \frac{x^3}{1+x^{\alpha} \sin^2 x}\, dx$$

converges if $\alpha>8$, and diverges if $\alpha \leq 8$.

Hence the results given in the question.

Question 800.

(S. Malhari Rao) :—Shew that the sum of all fractions which may be represented by a recurring decimal of the form $.a\,b\,\dot{c}\,\dot{d}$ is 50, provided

$$a+c=b+d=9.$$

Solution (1) *by G. L. Gupta, M.A. and S. V. Venkatachalayya.*

(i) For all values of a, b, c and d the recurring decimal $\cdot a\,b\,\dot{c}\,\dot{d}$

$$=\frac{abcd-ab}{9900}=\frac{(1000a+100b+10c+d)-(10a+b)}{9900}$$

$$=\frac{990a+99b+10c+d}{9900}. \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

But here $a+c=b+d=9.$

∴ (1) which may be written as

$$\frac{10(a+c)+(b+d)+98(10a+b)}{9900}$$

reduces to

$$\frac{99+98(10a+b)}{9900}=\frac{99+98(ab)}{9900} \quad \ldots \quad \ldots \quad (2)$$

But since each of a and b may have any of the values 0,1,2,3,...9 ab includes all integral numbers from 1 to 99, besides 0.

Hence the required sum

$$=\frac{1}{9900}\{(99\times100)+98(1+2+3+\ldots+99)\}$$

$$=\frac{1}{9000}\left\{9900+\frac{98\times90\times100}{2}\right\}=50.$$

Note:—There is a slight mistake in the printed question in which the decimal reads $(\cdot\ddot{a}bcd)$.

Solution by H. Br. and (2) *K. B. Madhava.*

(2) The recurring fraction $(.\dot{a}bc\dot{d})$

$$=\frac{1000a+100b+10c+d}{9999}$$

$$=\frac{99.(10a+b)+10(a+c)+(b+d)}{9999}$$

$$=\frac{10a+b+1}{101},\ \text{as}\ a+c=b+d=9,$$

As a and b assume independently all values from 0 to 9 (inclusive). the numerator assumes (once only) every integral values from 1 to 100, The sum of the fractions is therefore

$$\frac{1+2+3+\ldots\ldots+100}{101}=50.$$

Question 802.

(S. Krishnaswami Iyengar):—Prove that

(i) $$\sum_{1}^{\infty}\left\{\frac{\Gamma(n+1)}{\Gamma(n+\frac{3}{2})}\right\}^2\frac{1}{n}=\frac{4}{\sqrt{\pi}}\left(\frac{4}{\sqrt{\pi}}-\sqrt{\pi}\right),$$

(ii) $$\sum_{0}^{\infty}\frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)}\cdot\frac{1}{(2m+2n+1)}\cdot\frac{1}{(m+n+1)}$$

$$=\pi\left\{\frac{\Gamma(m+\frac{1}{2})}{\Gamma(m+1)}-\frac{2}{2m+1}\cdot\frac{\Gamma(m+1)}{\Gamma(m+\frac{1}{2})}\right\}.$$

Solution by S. R. Ranganathan and S. V. Venkatachallayya.

(i) This is the same as Q 664 solved on p. 69 of Vol. VIII, J.I.M.S.

(ii) We have, for $0\leq x\leq 1$,

$$(1-x)^{-\frac{1}{2}}=1+\tfrac{1}{2}x+\frac{1\cdot3}{2\cdot4}x^2+\ldots$$

so that,

$$\Gamma(\tfrac{1}{2})(1-x)^{-\frac{1}{2}}=\sum_{0}^{\infty}\frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)}x^n,$$

so that, $$\Gamma(\tfrac{1}{2})\int_0^1 x^m(1-x)^{-\frac{1}{2}}dx=\int_0^1\sum_0^\infty\frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)}x^{m+n}dx$$

$$=\sum_0^\infty\frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)}\int_0^1 x^{m+n}dx$$

$$=\sum\frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)}\frac{1}{m+n+1};$$

i.e., $$\{\Gamma(\tfrac{1}{2})\}^2\frac{\Gamma(m+1)}{\Gamma(m+\frac{3}{2})}=\sum\frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)}\frac{1}{m+n+1}.$$

In the above the change of order of integration and summation is permissible since the integral on the left is convergent and the terms in the series on the right are all positive (cf. Bromwich *Infinite Series* App. III, Art. 175. Theorem B.)

By treating similarly the expansion of $(1-x^2)^{-\frac{1}{2}}$ we get

$$\tfrac{1}{2}\{\Gamma(\tfrac{1}{2})\}^2\frac{\Gamma(m+\frac{1}{2})}{\Gamma(m+1)}=\sum_0^\infty\frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)}\frac{1}{(2m+2n+1)}$$

Now, $$\sum_0^\infty\frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)}\frac{1}{(2m+2n+1)(m+n+1)}$$

$$=\sum_0^\infty\frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)}\left\{\frac{2}{2m+2n+1}-\frac{1}{m+n+1}\right\}$$

$$=\{\Gamma(\tfrac{1}{2})\}^2\left\{\frac{\Gamma(m+\frac{1}{2})}{\Gamma(m+1)}-\frac{\Gamma(m+1)}{\Gamma(m+\frac{3}{2})}\right\}$$

$$=\pi\left\{\frac{\Gamma(m+\frac{1}{2})}{\Gamma(m+1)}-\frac{2}{(2m+1)}\frac{\Gamma(m+1)}{\Gamma(m+\frac{3}{2})}\right\}.$$

Question 810.

(T. P. TRIVEDI, M.A., L.L.B.) :—Find the values other than zero which satisfy the equations :—

$$x^2=y-z\ ;\ y^2=z-x\ ;\ z^2=x-y.$$

Solution by H. Br. S. Gangadharan, K. Santanam, Kasturi Reddi, B. Sc., and F. H. V. Gulasekkaran, B. A.

We have

$$\left.\begin{aligned} x^2 &= y-z \\ y^2 &= z-x \\ z^2 &= x-y. \end{aligned}\right\} \quad \ldots \quad (A)$$

$$\left.\begin{aligned} &\therefore \quad x^2+y^2+z^2=0\,; \\ &\qquad x^3+y^3+z^3=0\,; \\ &\text{and } x^4+y^4+z^4=\Sigma x^2(y-z)=-(y-z)(z-x)(x-y)=-x^2y^2z^2 \end{aligned}\right\} \quad B$$

[The system of equations B is composed of the system A together with the system derived from A by changing the signs of x, y, z.]

Let (x, y, z) be the roots of $t^3-pt^2+qt-r=0$. Then from (B),

$$\left.\begin{aligned} p^2 &= 2q, \\ pq &= 3r \\ \text{and} \qquad r(p+r) &= 0. \end{aligned}\right\}$$

If $r=0$, we get $p=q=0$, or $x=y=z=0$.

If $r\neq 0$, we get $q=-3$, $r=-p$; $p^2=-6$.

Thus x, y, z are the roots of

$$t^3-pt^2-3t+p=0, \text{ where } p^2=-6.$$

Put $\mu=pt+2$, and we get

$$\mu^3+6\mu+16=0,$$

The roots of this equation are

$$\mu_1=\sqrt{2}\ \{(3-2\sqrt{2})^{\frac{1}{3}}-(3+2\sqrt{2})^{\frac{1}{3}}\}$$

$$\mu_2=\sqrt{2}\{(3-2\sqrt{2})^{\frac{1}{3}}\omega-(3+2\sqrt{2})^{\frac{1}{3}}\omega^2\}$$

$$\mu_3=\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

From these we deduce the following types of values of x, y, z.

$$\left.\begin{aligned} 100x &= -1532\cdot \\ 100y &= -118-462\cdot \\ 100z &= 118-462\cdot \end{aligned}\right\} \text{approximately.}$$

QUESTIONS FOR SOLUTION.

943. (HEMRAJ) :—If n be prime and

$$(x+1)(x+2)\ldots(x+n-1) = x^{n-1} + A_1 x^{n-2} + A_2 x^{n-3} + \ldots A_{n-1}$$

shew that all the odd A's except A_1 are divisible by n^2—

944. (SADANAND) :—Prove the identity

$$\phi(x) + S_1\phi'(x).\frac{x}{1!} - S_2\phi''(x).\frac{x^3}{3!} + \ldots\ldots$$

$$\equiv \phi(2x) - S_2\phi''(2x)\,x^2 + S_4\phi''^1(2x)\,x^4 - \ldots\ldots$$

$$\frac{}{2!} \qquad\qquad \frac{}{4!}$$

where S_{2n} is the nth Eulerian number and S_{2n} is the n the" prepared Bernoullean number.

Deduce that $\log 2 = \frac{S_1}{1} - \frac{S_2}{2^3.1} - \frac{S_3}{3} + \frac{S_4}{2^5.2} + \frac{S_5}{5} \ldots\ldots$

945. (SADANAND) :—Fill up the vacant cells of the following magic square :—

46				16
	54		39	
		27		
	50		35	
73				43

946. (M. K. KEVALRAMANI) :—Show that there are forty points on the curve $x^2/a^2 + y^2/b^2 = 8x^2y^2$. Such that if tangents be drawn from them to the ellipse $x^2/a^2 + y^2/b^2 = 1$, the points of contact have got their eccentric angles in the ratio 1 : 2.

947. (M. K. KEVALRAMANI):—A triangle ABC inscribed in any lipse, touches a confocal ellipse at the points D, E, F respectively. Show that the ratio of the triangle DEF to ABC $= r/(2R)$, where r, R refer to the triangle ABC as usual.

948. (M. K. KEVALRAMANI):—A perfectly elastic particle, acted on by no forces is projected from the centre of a rectangle whose sides are $2a$ and $2b$ $(a>b)$ to strike the bigger side first and then goes on re-

bounding from its sides. If it ever pass through an angular point show that the direction of projection makes with the smaller side an angle $\tan^{-1}\left(\frac{a}{b}\frac{2n+1}{2m+1}\right)$ where m & n are integers, and if it ever strike any point of the smaller side other than the extremities, the angle must be of the form $\tan^{-1}\left(\frac{a}{b}\cdot\frac{2n+1}{2m}\right)$

949. (C. KRISHNAMACHARY):—With the usual notation for the numbers of Bernoulli and Euler, show that

$$2n = B_1\, 2^2\,(2^2-1)\binom{2n}{2} - B_2\, 2^4\,(2^4-1)\binom{2n}{4} + B_3\, 2^6\,(2^6-1)\binom{2n}{6} - \ldots + (-1)^{n-1} B_n\, 2^{2n}\,(2^{2n}-1).$$

$$(-1)^n E_n\,(2n+1) = (-1)^n B_n\, 2^{2n}\,(2^{2n}-1)\binom{2n+1}{1} + (-1)^{n-1} B_{n-1}\, 2^{2n-2}\,(2^{2n-2}-1)\binom{2n+1}{3} + \ldots - B_1\, 2^2\,(2^2-1)\binom{2n+1}{2} + (2n+1)$$

950. (C. KRISHNAMACHARY):—Collect the co-efficients of x^n in the series—

$$\frac{1}{1-x} - \frac{x}{(1-x)^2} + \frac{1\cdot3}{2!}\,\frac{x^2}{(1-x)^3} - \frac{1\cdot3\cdot5}{3!}\,\frac{x^3}{(1-x)^4} + \ldots \ldots$$

and show that its sum is zero when n is odd, and $(-1)^m \frac{1\cdot3\cdot5\cdot(2m-1)}{2\cdot4\cdot6\cdot\ldots\ 2m}$ when n is $2m$.

951. (C KRISHNAMACHARY):—Prove that

(1) $\int_0^{\frac{\pi}{4}} (\log\tan x)^{2n} dx = \frac{E_n}{2n+2}\,\pi^{2n+1}$,

(2) $\int_0^{\frac{\pi}{4}} \frac{(\log\tan x)^{2n-1}}{\cos x\,(\cos x+\sin x)}\,dx = -\frac{(2^{2n-1}-1)}{2n} B_n \pi^{2n}$,

(3) $\int_0^{\frac{\pi}{4}} \frac{(\log\tan x)^{2n-1}}{\cos 2x}\,dx = -\frac{(2^{2n}-1)B_n}{4n}\pi^{2n}$,

(4) $\int_0^{\frac{\pi}{4}} \frac{(\log\tan x)^{2n-1}}{\cos x\,(\cos x-\sin x)}\,dx = -\frac{2^{2n-1}B_n}{2n}\,\pi^{2n}$.

952. (Communicated by Mr. Hemraj) :—The polar reciprocal of an equiangular spiral with respect to a rectangular hyperbola having its centre at the pole and touching the spiral is the curve itself.

953. (Communicated by Mr. Hemraj) :—Two of the common tangents to a circle S and a conic T meet in P and the other two in Q. Show that P and Q lie on the same confocal to T.

954. (K. B. Madhava) :—When asked his age De Morgan once humorously observed : "I am one of those whose age shall be in a certain year belonging to the century of their birth the square root of that year." For, being born in 1806 A. D., he was 43 years old in 1849, and $\sqrt{(1849)}=43$. Show that the same observation will be true in the case of those born in 9 particular years in all subsequent centuries.

955. (K. B. Madhava) :—The 5th, 13th, 17th, 29th and 37th roots (if rational) of a number end in the same digit as the number itself, the numbers when fractional being expressed in decimals. Prove this property and illustrate how it enables one to give out at sight the above rational roots of big numbers.

956. (K. B. Madhava) :—Find the years of the present century in which the month of February will have (i) no full moon, (ii) no new moon.

957. (Martyn M. Thomas, M. A.) :—From a flexible envelope in the form of a surface of revolution formed by the curve $s=f(y)$ revolving about the axis of x, the part between two meridians the planes of which are inclined to each other at an angle $\frac{2\pi}{m}$ is cut away, and the edges are then sewed together. Prove that the meridian curve of the new envelope will be $s=f\left(\frac{m\,y}{m-1}\right)$.

Hence show that, if a lune of angle $\frac{4\pi}{5}$ be cut off from an oblate spheroid, the minor axis of whose generating ellipse is c, and eccentricity $\frac{3}{5}$, the meridian curve of the new surface of revolution will be the curve of sines $y=\frac{3}{4}c\sin\frac{x}{c}$.

958. (Martyn M. Thomas, M. A.,) :—Establish the formula

$$\frac{d^n u}{dx^n}=x^{r-n}\frac{d^r}{dx^r}\left[x^q\frac{d^{q-r}}{dx^{q-r}}\left\{x^{p-r}\frac{d^{p-q}}{dx^{p-q}}\left(x^{n-q}\frac{d^{n-p}}{dx^{n-p}}\cdot\frac{u}{x^p}\right)\right\}\right],$$

where $n>p>q>r$.

959. (M. T. Naraniengar) :—The sides AB, AC of a triangle ABC are produced to D, E such that $BD=CE=BC$.

If the circle ADE cuts the straight line AI_1 in J, prove that

$$I_1J=\tfrac{1}{2}\,II_1.$$

List of Periodicals Received.

(*From* 16*th January to* 15*th March* 1918.)

1. American Mathematical Monthly, Vol. 24, No. 8, October 1917.
2. Astrophysical Journal, Vol. 46, No. 3, October 1917.
3. Bulletin of the American Mathematical Society, Vol. 24, Nos. 2 and 3, November and December 1917.
4. Bulletin of the Calcutta Mathematical Society, Vol. 5 and 7 and 8, Nos. 1 and 2.
5. L'Intermediaire det Mathematiciens, Vol. 24 Nos. 5 to 12, May to December 1917.
6. Liouville's Journal, Vol. 2, No. 4.
7. Mathematical Gazette, Vol. 9, Nos. 131 and 132, October and December 1917.
8. Mathematical Questions and Solutions, Vol. 4, Nos. 6, December 1917. (5 Copies).
9. Mathematicet Koe Obrazovanie of Moteow. Nos. 1 to 42.
10. Messenger of Mathematics, Vol. 47, Nos. 2 and 3, June and July 1917.
11. Monthly notices of the Royal Astronomical Society, Vol. 77, No. 9, October 1917.
12. Notes on Elementry Mathematics, Nos. 1 to 21.
13. Philosophical Magazine, Vol. 34 and 35 Nos. 204 and 205, December and January 1918.
14. Popular Astronomy, Vol. 25, No. 9, November 1917. (3 Copies).
15. Proceedings of the Edinburgh Mathematical Society, Vol. 35, Part II.
16. Proceedings of the Royal Society of London, Vol. 94. No. 657, and 658.
17. School Science and Mathematics, Vol. 17, Nos. 8 and 9, November and December 1917.
18. Tohoku Mathematical Journal, Vol. 12, No. 4, November 1917.
19. Transaction of the American Mathematical Society, Vol. 18, No. 4, October 1917.
20. Transactions of the Royal Society of London, Vol. 217, Nos. 552, 553, 554, 555, 556, 557 and 559.

The following book has been purchased for the Library :—

A Synopsis of Pure Mathematics, by Mr. G. S. Carr, M A., 1888. Published by Messrs. Francis Hodgson, 89, Farringdon Street, E. C. London. Price £1-1s.

On the addition Formulae for the Jacobian Functions E and Π.

In the following pages I propose to deduce a few of the expressions for $Eu_1+Eu_2+Eu_3+Eu_4$ and those for $\Pi(u_1,a)+\Pi(u_2,a)+\Pi(u_3,a)+\Pi(u_4,a)$ in terms of the sn, cn, dn functions of u_1, u_2, u_3, u_4, when $u_1+u_2+u_3+u_4$ is congruent to zero.

§ I. Preliminary.

1. To begin with let us establish a few preliminary results which will be helpful to us in what follows.

For this purpose, let us consider the points of intersection of the fixed curve

$$y^2=x(1-x)(1-kx) \quad \ldots \quad \ldots \quad \ldots \quad \text{(i)}$$

with any arbitrary chosen parabola

$$y=l+mx+nx^2 \quad \ldots \quad \ldots \quad \ldots \quad \text{(ii)}$$

Now, if any point (x_1, y_1) be taken on the cubic (i), the equation in u

$$\text{sn}^2(u, k)-x_1=0, \text{ where } k \text{ is constant,}$$

has two solutions $+u_1, -u_1$, and all other solutions are congruent to these two.

Again since $\frac{\partial}{\partial u}(\text{sn}^2 u)=2\,\text{sn}\,u\,\text{cn}\,u\,\text{dn}\,u$, we have

$$\left[\frac{\partial}{\partial u}(\text{sn}^2 u)\right]^2=4y^2.$$

Let us choose u_1 to be the solution for which $\frac{\partial}{\partial u}(\text{sn}^2 u)=+2y_1$; and let (x_r, y_r), where $r=1, 2, 3, 4$, be the points of intersection of (i) and (ii); so that x_1, x_2, x_3, x_4 are the roots of the equation

$$F(x)=(l+mx+nx^2)^2-x(1-x)(1-k^2x)=0 \quad \ldots \quad \text{(iii)}$$

and

$$F(x)=(x-x_1)(x-x_2)(x-x_3)(x-x_4). \quad \ldots \quad \ldots \quad \text{(iv)}$$

The variation ∂x_r in one of these abscissae due to the variation in position of the parabola (ii) is given by the equation

$$F'(x_r)\partial x_r-2(l+m\,x_r+n\,x^2_r)(\partial l+x_r\partial m+x^2_r\partial n)=0.$$

Remembering that $l+m\,x_r+n\,x_r^2=y_r$, we obtain

$$\frac{\partial x_r}{y_r}=\frac{+2(\partial l+x_r\partial m+x^2_r n)}{F'(x_r)} \quad \ldots \quad \ldots \quad \text{(v)}$$

$$=\frac{f(x_r)}{F'(x_r)} \text{ suppose.}$$

Then since $xf(x)$ is of lower degree than $F(x)$,

$$\frac{xf(x)}{F(x)}=\sum_{r=1}^{4}\frac{f(x_r)}{F'(x_r)}\cdot\frac{x_r}{x-x_r} \quad \dots \quad \dots \text{ (vi)}$$

provided x_1, x_2, x_3, x_4 are all unequal, and $F'(x_r)\neq 0$.

Putting $x=0$, in (vi)

$$\sum_{r=1}^{4}\frac{f(x_r)}{F'(x_r)}=0.$$

$$\therefore \quad \sum_{r=1}^{4}\frac{\partial x_r}{y_r}=0.$$

$$\therefore \quad \sum_{r=1}^{4}\partial u_r=0.$$

Hence the sum of the parameters of the point of intersection is a constant and independent of the position of the parabola and it may be easily shewn to be congruent to zero.

Hence so long as $u_1+u_2+u_3+u_4=0$, we may deduce relations between symmetrical functions of s_r, c_r, d_r, (where s_r, c_r, d_r stand for sn u_r, cn u_r, dn u_r respectively) from the values of the symmetrical functions of the roots of the equation. ... ... (iii)

2. We have from equations (iii) and (iv) above,

$$\sum_{r=1}^{4}s^2_r=\sum_{r=1}^{4}x_r=\frac{k^2}{n^2}-2\frac{m}{n}. \quad \dots \quad \dots \text{ (α)}$$

$$s_1s_2s_3s_4=\sqrt{x_1x_2x_3x_4}=\frac{l}{n^2}. \quad \dots \quad \dots \quad \dots \text{ (β)}$$

$$c_1c_2c_3c_4=\sqrt{(1-x_1)(1-x_2)(1-x_3)(1-x_4)}=\frac{l+m+n}{n^2}. \quad \dots \text{ (γ)}$$

$$d_1d_2d_3d_4=\sqrt{(1-k^2x_1)(1-k^2x_2)(1-k^2x_3)(1-k^2x_4)}$$

$$=1+k^2\frac{m}{n}+k^4\frac{l}{n^2}. \quad \dots \quad \dots \quad \dots \text{ (δ)}$$

Hence eliminating l, m, n between (β), (γ) and (δ), we obtain the well known identity of Gudermann

$$k'^2-k^2k'^2s_1s_2s_3s_4+k^2c_1c_2c_3c_4-d_1d_2d_3d_4=0, \quad \dots \text{ (A)}$$

where $k'^2=1-k^2$.

Writing in the identity (A) successively, (u_3+K, u_4-K), (u_3-iK', u_4+iK'), (u_3+K-iK', u_4-K+iK) for (u_3, u_4) respectively, we obtain the following relations quoted in Whittaker and Watson's *Modern Analysis* (Revised Edition) from H. J. S. Smith, *Proceedings*, Lond. Math. Soc. (I) X

$$k^2(s_1s_2c_3c_4-c_1c_2s_3s_4)-d_1d_2+d_3d_4=0 \quad \ldots \quad \ldots \quad \text{(A1)}$$

$$k'^2(s_1s_2-s_3s_4)+d_1d_2c_3c_4-c_1c_2d_3d_4=0 \quad \ldots \quad \ldots \quad \text{(A2)}$$

$$s_1s_2d_3d_4-d_1d_2s_3s_4+c_3c_4-c_1c_2=0 \quad \ldots \quad \ldots \quad \ldots \quad \text{(A3)}$$

3. Again, let us evaluate the functions

$$\sum_{r=1}^{4}\frac{c_rd_r}{s_r},\ \sum_{r=1}^{4}\frac{s_rd_r}{c_r},\ \sum_{r=1}^{4}\frac{s_rc_r}{d_r} \quad \text{in terms of } l, m, n.$$

We have

$$\sum_{r=1}^{4}\frac{c_rd_r}{s_r}=\sum_{r=1}^{4}\frac{y_r}{x_r}=\sum_{r=1}^{4}\frac{l+mx_r+nx_r^2}{x_r}$$

$$=\Sigma\frac{1}{x_r}+n\Sigma x_r+4m,$$

$$=\frac{1}{l}+\frac{k^2}{n}.$$

Again $$\sum\frac{s_rd_r}{c_r}=\sum\frac{y_r}{1-x_r}=\sum\frac{l+mx_r+nx_r^2}{1-x_r}$$

This function is best evaluated by substituting p_r for $1-x_r$ and forming the equation whose roots are the four values of p_r. We can then by easy algebraical processes deduce that

$$\sum\frac{s_rd_r}{c_r}=\frac{k^2}{n}-\frac{k'^2}{l+m+n}.$$

Similarly from the equation whose roots are $1-kx_r$, we deduce that

$$\sum\frac{s_rc_r}{d_r}=\frac{lk^4+mk^2+n(1+k'^2)}{n(n+mk^2+lk^4)}.$$

From all the foregoing, we are in a position to deduce several relations. We shall select only such as will be useful for our purpose.

4. We have

$$\frac{1}{n}=\frac{1}{k}\sqrt{(\Sigma s_r^2-2s_1s_2s_3s_4+2c_1c_2c_3c_4-2)} \quad \ldots \quad \ldots \quad \text{(B1)}$$

$$=\sqrt{(2-\Sigma d_r^2+2d_1d_2d_3d_4-2k^4s_1s_2s_3s_4)} \quad \ldots \quad \ldots \quad \text{(B2)}$$

$$=\frac{-1}{k^2+k'^2(c_1c_2c_3c_4)^{-1}}\sum\frac{c_rd_r}{s_r} \quad \ldots \quad \ldots \quad \ldots \quad \text{(B3)}$$

$$=\frac{1}{k^2-k'^2(c_1c_2c_3c_4)^{-1}}\sum\frac{s_rd_r}{c_r} \quad \ldots \quad \ldots \quad \ldots \quad \text{(B4)}$$

$$=\frac{1}{1+k'^2(d_1d_2d_3d_4)^{-1}}\sum\frac{s_rc_r}{d_r} \quad \ldots \quad \ldots \quad \ldots \quad \text{(B5)}$$

Further, writing $u_r+\mathrm{K}$ for u_r in B3 and in B4, and $u_r-i\,\mathrm{K}'$ for u_r in B5, we obtain

$$\frac{1}{n}=\frac{-1}{(c_1c_2c_3c_4)^{-1}+k^2(d_1d_2d_3d_4)^{-1}\sum\frac{s_r}{c_rd_r}} \quad \ldots \quad \ldots \quad \text{(B6)}$$

$$=\frac{1}{k^2k'^2(d_1d_2d_3d_4)^{-1}-(s_1s_2s_3s_4)^{-1}\sum\frac{c_r}{s_rd_r}} \quad \ldots \quad \ldots \quad \text{(B7)}$$

$$=\frac{k^2s_1s_2s_3s_4+c_1c_2c_3c_4}{k'^2s_1s_2s_3s_4+c_1c_2c_3c_4\sum\frac{d_r}{s_rc_r}} \quad \ldots \quad \ldots \quad \ldots \quad \text{(B8)}$$

the summation extending to the suffixes 1, 2, 3, 4.

By eliminating x between the equations (i) and (ii) of § I, we obtain the equation whose roots are y_1, y_2, y_3, y_4. Evaluating from this equation $\sum_{r=1}^{4}\frac{1}{y^r}$, we can easily prove

$$\frac{1}{n}=-\left\{(s_1s_2s_3s_4)^{-1}+(c_1c_2c_3c_4)^{-1}+k^4(d_1d_2d_3d_4)^{-1}\right\}^{-1}\sum_{r=1}^{4}\frac{1}{s_rc_rd_r} \qquad \text{(B9)}$$

§ II. $\Sigma\,\mathrm{E}(u_r, k)$ when $u_1+u_2+u_3+u_4=0$.

5. We have $\mathrm{E}(u_r, k)=\int(\mathrm{dn}^2u_r)\,\partial u_r$

$$=u_r-k^2\int(\mathrm{sn}^2u_r)\partial u_r$$

$$=u_r-\frac{1}{2}k^2\int\frac{x_r\,\partial x_r}{y_r}.$$

$$\therefore \quad \sum_{r=1}^{4}\mathrm{E}(u_r, k)=-\frac{1}{2}k^2\sum_{r=1}^{4}\int\frac{x_r\,\partial x_r}{y_r}. \quad \ldots \quad \ldots \quad (\Sigma)$$

Now going back to our equation (v) of § I

$$x_r\frac{\delta x_r}{y_r}=\frac{+2x_r\,(\partial l+x_r\partial m+x_r^2\,\partial n)}{\mathrm{F}'(x_r)}.$$

By the method of partial fractions, we can easily prove that

$$\sum_{r=1}^{4}\frac{x_r}{\mathrm{F}'(x_r)}=0,\quad \sum_{r=1}^{4}\frac{x_r^2}{\mathrm{F}'(x_r)}=0 \quad \text{and} \quad \sum_{r=1}^{4}\frac{x_r^3}{\mathrm{F}'(x_r)}=+\frac{1}{n^3}.$$

Hence $$\sum_{r=1}^{4}\frac{x_r\,\partial x_r}{y_r}=\frac{2}{n^3}\partial n.$$

$$\therefore \quad \sum_{r=1}^{4}\int\frac{x_r\,\partial x_r}{y_r}=-\frac{2}{n}.$$

$$\therefore \text{ from } (\Sigma), \sum_{r=1}^{4} \mathrm{E}(u_r, k) = k^2\left(\frac{1}{n}\right). \quad \ldots \quad \ldots \quad \ldots \quad (C)$$

Hence $\sum_{r=1}^{4} \mathrm{E}(u_r, k)$ has for its values in terms of s_r, c_r, d_r each of the expressions in B 1—B 9 in § I multiplied by k^2.

6. $\mathrm{E}(u_1,k)+\mathrm{E}(u_2, k)+\mathrm{E}(u_3, k)$ when $u_1+u_2+u_3=0$.

In the above, putting $u_4=0$, we deduce that when $u_1+u_2+u_3=0$,

$\mathrm{E}(u_1, k)+\mathrm{E}(u_2, k)+\mathrm{E}(u_3, k)$

$$=k\sqrt{s_1^2+s_2^2+s_3^2+2c_1c_2c_3-2} \quad \ldots \quad \ldots \quad \ldots \quad (D1)$$

$$=k\sqrt{1-c_1^2-c_2^2-c_3^2+2c_1c_2c_3} \quad \ldots \quad \ldots \quad \ldots \quad (D2)$$

$$=\sqrt{(1-d_1^2-d_2^2-d_3^2+2d_1d_2d_3)} \quad \ldots \quad \ldots \quad \ldots \quad (D3)$$

$$=-k^2s_1s_2s_3 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (D4)$$

$$=\frac{k^2}{k^2-k'^2(c_1c_2c_3)^{-1}}\left\{\frac{s_1d_1}{c_1}+\frac{s_2d_2}{c_2}+\frac{s_3d_3}{c_3}\right\} \quad \ldots \quad \ldots \quad (D5)$$

$$=\frac{k^2}{1+k'^2(d_1d_2d_3)^{-1}}\left\{\frac{s_1c_1}{d_1}+\frac{s^2c_2}{d_2}+\frac{s_3c_3}{d_3}\right\} \quad \ldots \quad \ldots \quad (D6)$$

§ III. $\Sigma II(u_r, k, a)$ when $u_1+u_2+u_3+u_4=0$.

7. Now, since

$$II(u_r, k, a)=\int \frac{k^2 \mathrm{sn}a\, \mathrm{cn}a\, \mathrm{dn}a\, \mathrm{sn}^2 u_r}{1-k^2\mathrm{sn}\, a\, \mathrm{sn}^2 u_r}\partial u_r$$

$$\therefore \quad II(u_r, k, a)+\frac{\mathrm{cn}\, a\, \mathrm{dn}\, a}{\mathrm{sn}\, a}.u_r=\frac{\mathrm{cn}a\, \mathrm{dn}\, a}{\mathrm{sn}\, a}\int \frac{\partial u_r}{1-k^2\mathrm{sn}^2 a\, \mathrm{sn}^2 u_r}.$$

Then putting $\mathrm{sn}^2 u_r=\varpi_r$; $\frac{1}{k^2\mathrm{sn}^2 a}=\lambda$; $\frac{\mathrm{cn}\, a\ \mathrm{dn}\, a}{k^2\mathrm{sn}^3 a}=\mu$; and remembering that $\sum_{r=1}^{4} u_r=0$, we easily deduce that

$$\sum_{r=1}^{4} II(u_r, k, a)=\tfrac{1}{2}\mu \sum_{r=1}^{4}\int\left\{\frac{1}{\lambda-x_r}\frac{\partial x_r}{y_r}\right\}. \quad \ldots \quad (\omega)$$

Again going back to equation (v̇) of § I,

$$\frac{\partial x_r}{y_r}=\frac{+2(\delta l+x_r\delta m+x_r^2\delta n)}{F'(x_r)}=\frac{f(x_r)}{F'(x_r)} \text{ (suppose).}$$

$$\sum_{r=1}^{4}\frac{1}{\lambda-x_r}\frac{\partial x_r}{y_r}=\sum_{r=1}^{4}\frac{f(x_r)}{(\lambda-x_r)F'(x_r)}.$$

Again from (iv) of § 1,

$$F(\lambda)=n^2(\lambda-x_1)(\lambda-x_2)(\lambda-x_3)(\lambda-x_4)$$

$$\therefore \qquad \sum\frac{f(\lambda)}{F(\lambda)}=\sum_{r=1}^{4}\frac{f(x_r)}{(\lambda-x_r)F'(x_r)}$$

since, $f(\lambda)$ is of lower degree than $F(\lambda)$,

$$\text{Hence } \sum_{r=1}^{4}\frac{1}{\lambda-x_r}\frac{\partial x_r}{y_r}=\frac{f(\lambda)}{F(\lambda)}=\frac{+2\partial(l+m\lambda+n\lambda^2)}{(l+m\lambda+n\lambda^2)^2-\mu^2}$$

$$=\frac{1}{\mu}\left[\frac{\partial(l+m\lambda+n\lambda^2)}{l+m\lambda+n\lambda^2-\mu}-\frac{\partial(l+m\lambda+n\lambda^2)}{l+m\lambda+n\lambda^2+\mu}\right]$$

$$=\frac{1}{\mu}\partial\log\frac{l+m\lambda+n\lambda^2-\mu}{l+m\lambda+n\lambda^2+\mu}$$

$$\therefore \qquad \mu\sum_{r=1}^{4}\int\frac{1}{\lambda-x_r}\cdot\frac{\partial x_r}{y_r}=\log\frac{l+m\lambda+n\lambda^2-\mu}{l+m\lambda+n\lambda^2+\mu}.$$

$$\therefore \qquad \sum_{r=1}^{4}II(u_r,k,a)=\frac{1}{2}\log\frac{l+m\lambda+n\lambda^2-\mu}{l+m\lambda+n\lambda^2+\mu}.$$

$$=\tfrac{1}{2}\log\Phi, \text{ say.} \qquad \ldots \qquad \ldots \text{ (H)}$$

After substituting for λ and μ,

$$\Phi=\frac{1+\frac{m}{n}k^2\mathrm{sn}^2a+\frac{l}{n}k^4\mathrm{sn}^4a-\frac{k^2}{n}\,\mathrm{sn}a\,\mathrm{cn}a\,\mathrm{dn}a}{1+\frac{m}{n}k^2\mathrm{sn}^2a+\frac{l}{n}k^4\mathrm{sn}^4a+\frac{k^2}{n}\,\mathrm{sn}a\,\mathrm{cn}a\,\mathrm{dn}a}$$

Hence after substituting from (β), (γ) of § I and from (C) of § II respectively,

$$\Phi=\frac{1+P\,\mathrm{sn}^2a+Q\,\mathrm{sn}^4a-R\,\mathrm{sn}a\,\mathrm{cn}a\,\mathrm{dn}a}{1+P\,\mathrm{sn}^2a+Q\,\mathrm{sn}^4a+R\,\mathrm{sn}a\,\mathrm{cn}a\,\mathrm{dn}a},$$

where

$$P=k^2(c_1c_2c_3c_4-s_1s_2s_3s_4-1)=(d_1d_2d_3d_4-k^4s_1s_2s_3s_4-1)$$

$$=\frac{1}{k'^2}(d_1d_2d_3d_4-k^4c_1c_2c_3c_4-1+k^4),$$

$$Q=k^4s_1s_2s_3s_4,$$

$$R=\sum_{r=1}^{4}E(u_r,k),$$

which is already expressed in terms of s_r, c_r, d_r in § II.

8. Now putting $u_4=0$, we have from above, when $u_1+u_2+u_3=0$,

$$II\,(u_1,k,a)+II(u_2,k,a)+II(u_3,k,a)$$
$$=\tfrac{1}{2}\log\frac{1+\mathrm{U}\,\mathrm{sn}^2a-\mathrm{W}\,\mathrm{sn}a\ \mathrm{cn}a\ \mathrm{dn}a}{1+\mathrm{U}\,\mathrm{sn}^2a+\mathrm{W}\,\mathrm{sn}a\ \mathrm{cn}a\ \mathrm{dn}a}=\tfrac{1}{2}\log\Omega\ (\text{say}) \qquad \ldots \quad (\mathrm{K})$$

where $\mathrm{U}=k^2(c_1c_2c_3-1)=d_1d_2d_3-1$
$$=k^2[s_1s_2c_3d_3-s_3^2]$$
$$\mathrm{W}=\mathrm{E}(u_1,k)+\mathrm{E}(u_2,k)+\mathrm{E}(u_3,k)$$
$$=\text{the expressions D1—D6 in § II.}$$
$$=-k^2s_1s_2s_3 \text{ in particnalr.}$$

With these values

$$\Omega=\frac{\mathrm{dn}^2a+k^2\mathrm{sn}^2a\ c_1c_2c_3+k^2\mathrm{sn}a\ \mathrm{cn}a\ \mathrm{dn}a.s_1s_2s_3}{\mathrm{dn}^2a+k^2\mathrm{sn}^2a\ c_1c_2c_3-k^2\mathrm{sn}a\ \mathrm{cn}a\ \mathrm{dn}a.s_1s_2s_3} \qquad \ldots \quad (\mathrm{K1})$$

$$=\frac{\mathrm{cn}^2a+\mathrm{sn}^2a\ d_1d_2d_3+k^2\mathrm{sn}a\ \mathrm{cn}a\ \mathrm{dn}a\ s_1s_2s_3}{\mathrm{cn}^2a+\mathrm{sn}^2a.d_1d_2d_3-k^2\mathrm{sn}a\ \mathrm{cn}a\ \mathrm{dn}a\ s_1s_2s_3} \qquad \ldots \quad (\mathrm{K2})$$

expressions easily remembered.

9. It will be interesting to deduce from the above the expressions for Ω given in Dixon's *Elliptic Functions*.

Taking $\mathrm{U}=k^2(s_1s_2c_3d_3-s_3^2)$

and $\mathrm{W}=-k^2s_1s_2s_3$,

the expression (K) above reduces to

$$\Omega=\frac{(1-k^2\mathrm{sn}^2a\mathrm{sn}^2u_3)+k^2\mathrm{sn}a.s_1s_2(\mathrm{sn}a.c_3d_3+s_3.\mathrm{cn}a\ \mathrm{dn}a)}{(1-k^2\mathrm{sn}^2a\mathrm{sn}^2u_3)+k^2\mathrm{sn}a.s_1s_2(\mathrm{sn}a.c_3d_3-s_3.\mathrm{cn}a\ \mathrm{dn}a)}$$

dividing the numerator and denominator by $1-k^2\mathrm{sn}^2a\mathrm{sn}^2u_3$, we obtain

$$\Omega=\frac{1+k^2\mathrm{sn}a.s_1s_2\mathrm{sn}(a+u_3)}{1+k^2\mathrm{sn}a.s_1s_2\mathrm{sn}(a-u_3)}. \qquad \ldots \quad \ldots \quad \ldots \quad (\mathrm{M1})$$

10. Again from the expressions for $\mathrm{sn}\,u_1\,\mathrm{sn}\,u_2$, $\mathrm{sn}\,a\,\mathrm{sn}\,(u_1+u_2-a)$ obtained by substituting in turn $\mathrm{A}=u_1$, $\mathrm{B}=u_2$; $\mathrm{A}=u_1+u_2-a$, $\mathrm{B}=a$ in the formula

$$\mathrm{snA\ snB}=\frac{\mathrm{sn}^2\tfrac{1}{2}(\mathrm{A}+\mathrm{B})-\mathrm{sn}^2\tfrac{1}{2}(\mathrm{A}-\mathrm{B})}{1-k^2\mathrm{sn}^2\tfrac{1}{2}(\mathrm{A}+\mathrm{B})\mathrm{sn}^2\tfrac{1}{2}(\mathrm{A}-\mathrm{B})}, \qquad \ldots \quad \ldots \quad (\mathrm{N})$$

we have

$$1-k^2\mathrm{sn}\,u_1\,\mathrm{sn}\,u_2\,\mathrm{sn}(u_1+u_2-a)=$$
$$\left\{\frac{1-k^2\mathrm{sn}^2\tfrac{1}{2}(u_1+u_2)}{1-k^2\mathrm{sn}^2\tfrac{1}{2}(u_1+u_2)\mathrm{sn}^2\tfrac{1}{2}(u_1-u_2)}\right\}\left\{\frac{1-k^2\mathrm{sn}^2\tfrac{1}{2}(u_1-u_2)\mathrm{sn}^2\tfrac{1}{2}(u_1+u_2-2a)}{1-k^2\mathrm{sn}^2\tfrac{1}{2}(u_1+u_2)\mathrm{sn}^2\tfrac{1}{2}(u_1+u_2-2a)}\right\}.$$

Writing $-a$ for a in this equation, we have a second equation, which divided by the first gives

$$\Omega=\left\{\frac{1-k^2\mathrm{sn}^2\tfrac{1}{2}(u_1-u_2)\mathrm{sn}^2\tfrac{1}{2}(u_1+u_2-2a)}{1-k^2\mathrm{sn}^2\tfrac{1}{2}(u_1-u_2)\mathrm{sn}^2\tfrac{1}{2}(u_1+u_2+2a)}\right\}\times$$
$$\left\{\frac{1-k^2\mathrm{sn}^2\tfrac{1}{2}(u_1+u_2)\mathrm{sn}^2\tfrac{1}{2}(u_1+u_2+2a)}{1-k^2\mathrm{sn}^2\tfrac{1}{2}(u_1+u_2)\mathrm{sn}^2\tfrac{1}{2}(u_1+u_2-2a)}\right\}. \qquad (\mathrm{M2})$$

Again from the formula (N) above,

$$1 \pm k \operatorname{sn} A \operatorname{sn} B = \frac{\{1 \pm k \operatorname{sn}^2 \tfrac{1}{2}(A+B)\}\ \{1 \mp k \operatorname{sn}^2 \tfrac{1}{2}(A-B)\}}{1-k^2 \operatorname{sn}^2 \tfrac{1}{2}(A+B) \operatorname{sn}^2 \tfrac{1}{2}(A-B)}.$$

$$\therefore \quad [1-k^2 \operatorname{sn}^2 \tfrac{1}{2}(A+B) \operatorname{sn}^2 \tfrac{1}{2}(A-B)]^2 = \frac{\{1-k^2 \operatorname{sn}^4 \tfrac{1}{2}(A+B)\}\ \{1-k^2 sn^4 \tfrac{1}{2}(A-B)\}}{(1-k^2 \operatorname{sn}^2 A \operatorname{sn}^2 B)}.$$

In this equation writing in turn

$$A = u_1 + a,\ B = u_2 + a;$$
$$A = u_1 - a,\ B = u_2 - a;$$
$$A = u_1 + u_2 - a,\ B = a;$$
$$A = u_1 + u_2 + a,\ B = a;$$

we get expressions for the square of each of the factors of the numerator and denominator of the expression M2.

Hence we have after cancelling like factors in the numerator and denominator,

$$\Omega^2 = \frac{\{1-k^2 \operatorname{sn}^2 (u_1+a) \operatorname{sn}^2 (u_2+a)\}}{\{1-k^2 \operatorname{sn}^2 (u_1-a) \operatorname{sn}^2 (u_2-a)\}} \times \frac{\{1-k^2 \operatorname{sn}^2 a \operatorname{sn}^2 (u_1+u_2-a)\}}{\{1-k^2 \operatorname{sn}^2 a \operatorname{sn}^2 (u_1+u_2+a)\}}. \qquad \ldots \quad (M2)$$

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SHORT NOTES.

Pascal Hexagon.

The concurrence of the Pascal lines of a hexagon inscribed in a conic, is proved here by the properties of the Homographic Function. *Reference*: The Note at the end of Salmon's *Conics* and my previous paper on "the Homographic Function as an Operator." (J. I. M. S, vol. VIII, p. 202),

I. *Introductory*: The properties connected with Pascal's Hexagon deal with cases in which three correspondences have their double points in involution, No simple necessary conditions for this can be given, but the following theorems give sufficient conditions and are applicable in many cases.

Theorem I. *If S is a given correspondence and S_1 a variable correspondence with given double points, then the double points of $S_1 S$ belong to a fixed involution.*

Let I represent the involution determined by the double points of S_1 and S. Then S_1 can be decomposed into the product of a variable involution I_1 containing its double points and the involution I (J.I.M.S. vol. VIII, p. 202)

Thus $S_1 S = I_1 I \cdot S$.

Again since I contains the double points of S, IS is a fixed involution I′ containing the double points of S—(*Ibid*)

Hence $$S_1 S = I_1 I_1.$$

The double points of $S_1 S$ are the double points of $I_1 I'$ *i.e.*, the common pair of I_1 and I′. Thus the double points S_1 S belong to the fixed involution I′.

Cor. In particular if Ω be a correspondence of period three, the double points of S, Ω S, Ω^2 S, belong to an involution.

Theorem II. *If I_1, I_2, I_3, be three involutions having a common pair, the double points of $I_1 S$, $I_2 S$, $I_3 S$, belong to an involution.*

Denote the common pair by i, and the double pair of S by s' Then by the theorem quoted above I_1 I_2 I_3 can each be split up into the product of the involution I containing i and s and three correspondences having p as their double pair. Hence putting IS=S′, the theorem is reduced to Theorem I.

Theorem III. *If the product of three correspondences is an involution in whichever order they are taken, then their double points belong to an involution.*

If $S_1S_2S_3$ and $S_2S_1S_3$ are involutions, the product of S_1, S_2, S_3 in any order is an involution. (This follows from the theorem that the parameter of the product of a number of correspondences depends not on the actual but only on the cyclical order of the product. (*vide*: the paper on one-one correspondence above referred to)

The condition that the correspondence $\frac{ax+b}{cx+d}$ may be an involution, is $a+d=0$.

Hence forming the products $S_1S_2S_3$, $S_2S_1S_3$ by the ordinary rules of multiplication of matrices, we have, for the condition that these may be involutions, the equations

$$(a_1a_2+b_1c_2)a_3+(a_1b_2+b_1d_2)c_3+(c_1a_2+d_1c_2)b_3+(c_1b_2+d_1d_2)d_3=0.$$

and $(a_1a_2+b_2c_1)a_3+(a_2b_1+b_2d_1)c_3+(c_2a_1+d_2c_1)b_3+(c_2b_1+d_1d_2)d_3=0.$

Subtracting we find

$$\begin{vmatrix} c_1 & d_1-a_1 & b_1 \\ c_2 & d_2-a_2 & b_2 \\ c_3 & d_3-a_3 & b_3 \end{vmatrix} =0,$$

which shews that the double points of S_1, S_2, S_3 belong to an involution.

II. *Properties of three Points*:

Three points pqr determine two periodic correspondences

$$\begin{vmatrix} pqr \\ qrp \end{vmatrix} \text{ and } \begin{vmatrix} prq \\ rqp \end{vmatrix}.$$

If Ω denote one of these, the other is Ω^2. The double points of Ω will be called the *cyclic centres* of pqr. (These must be distingnished from the so-called harmonic centres. The cyclic centres are defined by a purely descriptive process while the harmonic centres, as the name *does not* indicate, are defined by a metrical property.)

(1). From the definition of the cyclic centres as the double points of the correspondence $\begin{vmatrix} pqr \\ qrp \end{vmatrix}$, it follows that each of the involutions (pp,qr) (qq,rp), (rr,pq) contain the cyclic centres.

(2) *Geometrical Interpretation.*

Let PQR be points on the fundamental conic corresponding to pqr. Let the tangents at PQR meet the opposite sides in P′,Q′,R′, Then P′ represents the double pair of the involution (pp,qr). Hence (1) proves that P′Q′R′ are collinear. The intersections of P′Q′R′ with the conic correspond to the cyclic centres of pqr.

(3) Theorem IV. *If a correspondence carries* (pqr) *into* $(p'q'r')$, *it carries the cyclic centres of* (pqr) *into the cyclic centres of* $p'q'r'$.

Let $S=\begin{vmatrix} p & q & r \\ p' & q' & r' \end{vmatrix}$, $\Omega=\begin{vmatrix} q & r & p \\ p & q & r \end{vmatrix}$, $\Omega^1=\begin{vmatrix} p'q'r' \\ q'r'p' \end{vmatrix}$.

Then we see that $\Omega'S\Omega=S$, identically.

Hence if x be a double point of Ω

$$S(x)=\Omega' S\Omega(x)=\Omega' S(x)\ ;$$

so that S (x) is a double point of Ω'; which proves the theorem, since the double points of Ω and Ω' are the cyclic centres of (pqr) and $(p'q'r')$ respectively.

III. *Six Points*: Six points $pqr\ p'q'r'$ can be divided into two groups of three, in 10 ways. From each division we can get six correspondences by keeping one group in a fixed order and permuting the other. Hence we get sixty correspondences (a correspondence and its inverse not being regarded as distinct).

If the six points are represented as the vertices of a hexagon inscribed in the fundamental conic, the double points of these sixty correspondences are the intersections with the conics of the sixty Pascal lines.

(1) We first shew that the double points of the six correspondences which can be derived from the same division, belong three and three to two involutions.

Consider for example, the division $(pqr)\ (p'q'r')$.

$$\text{Let } S_1=\begin{vmatrix} p & q & r \\ p' & q' & r' \end{vmatrix},\ S_2=\begin{vmatrix} p & q & r \\ q' & r' & p' \end{vmatrix},\ S_3=\begin{vmatrix} p & q & r \\ r' & p' & q' \end{vmatrix}.$$

$$S_1'=\begin{vmatrix} p & q & r \\ p' & r' & q' \end{vmatrix},\ S_2'=\begin{vmatrix} p & q & r \\ r' & q' & p' \end{vmatrix},\ S_3'=\begin{vmatrix} p & q & r \\ q' & p' & r' \end{vmatrix}.$$

If Ω, Ω^2 be the two periodic correspondences which carry the group $(p'q'r')$ into itself, we have evidently

$$S_1=S_1,\ S_2=\Omega S_1,\ S_3=\Omega^2 S_1,$$
$$S_1'=S_1',\ S_2'=\Omega^2 S_1',\ S_3=\Omega S_1'.$$

Hence, by Th. I, Cor., the double points of $S_1S_2S_3$ belong to an involution I and the double points of S_1', S_2', S_3' belong to another involution I'.

(2) *Each of the involutions I, I' contains the double points of the other*

Let $(t_1t_2), (t_1't_2')$ be the cyclic centres of (pqr) and $(p'q'r')$ respectively. Every one of the six correspondences we are considering carries the group (t_1t_2) into the group $(t_1't_2')$ (Th. IV.) Hence if i, i'' represent the involutions $(t_1t_1',\ t_2t_2')$ and $(t_1t_2',t_1't_2)$ respectively, the double points of every one of the six correspondences, must belong either to i or to i''.

Hence the double points of two at least of the three correspondences S_1, S_2, S_3 must belong to one of the involutions i, i'' say i and therefore the double points of the third also must belong to i (for we have proved that the double points of S_1, S_2, S_3 belong to an involution).

Hence I must be the same as i.

Similarly it is seen that I' in the same as i''.

Now the product of i' and i'' is an involution, viz. $(t_1t_2,t_1't_2')$. Hence (Th. II, Cor. in the paper mentioned above) i and i'' each contain the double points of the other. This is seen otherwise also, for i, i'' are represented in the plane by two sides of the harmonic triangle of the quadrangle $t_1t_2t_1't_2'$.

Thus the involutions determined by the double points of S_1, S_2, S_3 and S_1',S_2',S_3' each contains the double points of the other.

Hence we see that *the sixty Pascal lines meet three by three in* 20 *points* (*Steiner's points which form ten pairs of conjugate points w.r.t the conic, each conjugate pair corresponding to a division of the six points.* (The latter part is mentioned but not proved in Salmon).

(3) *Kirkman's Points.*

Consider the correspondences

$$S_1=\begin{vmatrix} p & qr' \\ q'p'r \end{vmatrix},\ S_2=\begin{vmatrix} q & rp' \\ r'q'p \end{vmatrix},\ S_3=\begin{vmatrix} rp & q' \\ p'r' & q \end{vmatrix}.$$

We have

$$S_1S_2S_3(q')=r$$
$$S_1S_2S_3(r)=q';$$

hence $S_1S_2S_3$ is an involution.

Again
$$S_1S_3S_2\ (p')=r$$
$$S_1S_3S_2\ (r)=p';$$

hence $S_1S_3S_2$ is an involution.

Thus every product of S_1, S_2 and S_3 is an involution and therefore the double points of S_1, S_2 and S_3 belong to an involution. (Theorem III.)

Thus the Pascal lines corresponding to S_1, S_2, S_3 meet in the point (Kirkman's point) which represents the double pair of this involution. In the present case S_2, S_3 are got from S_1 by permuting cyclically two points in the upper group of S_1 and the non-corresponding point in the lower group and also permuting cyclically the corresponding other points in the opposite sense.

There are three ways in which this double permutation can be made.

Thus three types of correspondences S_2S_3 can be obtained from S_1.

Hence every Pascal line contains 3 Kirkman points (and one Steiner's point.)

There are therefore $\frac{60\times3}{3}$ or 60 Kirkman points.

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Note on the Gamma Function.

Gauss' Formula.

For n a positive integer $\Gamma(n)=\int_0^\infty e^{-x}x^{n-1}dx=(n-1)!$

$$\Gamma(a)=\int_0^\infty e^{-x}\,x^{a-1}\,dx=\frac{1}{a(a+1)\ldots(a+n)}\int_0^\infty x^{a+n}\,e^{-x}\,dx, \text{ where } a>0$$

$$=\frac{1\cdot2\cdot3\ldots n}{a(a+1)\ldots(a+n)}\cdot\frac{\int_0^\infty x^{a+n}\,e^{-x}\,dx}{\int_0^\infty x^{n}\,e^{-x}\,dx}=\frac{1\cdot2\cdot3\ldots n}{a(a+1)\ldots(a+n)}n^a\cdot\frac{\int_0^\infty x^{a+n}\,e^{-nx}\,dx}{\int_0^\infty x^{n}\,e^{-nx}\,dx}$$

It can now be shown that the ratio of the two integrals tends to 1 as n tends to infinity.

Suppose a to be between r and $r+1$, where r is a positive integer, then

$$\int_1^\infty x^{r+n}\,e^{-nx}\,dx<\int_1^\infty x^{a+n}\,e^{-nx}\,dx<\int_1^\infty x^{r+1+n}\,e^{-nx}dx; \quad \ldots \quad (1)$$

also $\int_0^1 x^{a+n}e^{-nx}\,dx$ lies between 0 and $\int_0^1 x^{a+n}\,dx$, or $\frac{1}{a+n+1}$,

whence we have

$$\int_0^1 x^{a+n}\,e^{-nx}\,dx=\frac{\theta}{a+n+1}, \text{ where } 0<\theta<1.$$

Hence from the inequalities (1) we can write

$$\frac{\frac{\theta}{a+n+1}-\frac{\theta'}{r+n+1}+\int_0^\infty x^{r+n}\,e^{-nx}\,dx}{\int_0^\infty x^{n}\,e^{-nx}\,dx}<\frac{\int_0^\infty x^{a+n}\,e^{-nx}\,dx}{\int_0^\infty x^{n}\,e^{-nx}\,dx}$$

$$<\frac{\frac{\theta}{a+n+1}-\frac{\theta''}{r+n+2}+\int_0^\infty x^{r+1+n}\,e^{-nx}\,dx}{\int_0^\infty x^{n}\,e^{-nx}\,dx}.$$

Consider $\dfrac{\int_0^\infty x^{r+n}\,e^{-nx}\,dx}{\int_0^\infty x^{n}\,e^{-nx}\,dx}=\dfrac{1}{n^r}\cdot\dfrac{1\cdot2\cdot3\ldots(r+n)}{1\cdot2\cdot3\ldots n}$

$$=\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\ldots\left(1+\frac{r}{n}\right);$$

since there are only a finite number of factors the limit as n tends to infinity of the ratio of the integrals tends to 1, and hence

$$\Gamma(a)=\lim_{n\to\infty}\frac{1\cdot2\cdot3\ldots n}{a(a+1)\ldots(a+n)}n^a.$$

2. *The relation between the Beta and Gamma Functions.*

$$\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}=\lim_{\mu\to\infty}\frac{1\cdot2\cdot3\ldots\mu}{n(n+1)\ldots(n+\mu)}\cdot\frac{(m+n)\ldots(m+n+\mu)}{m(m+1)\ldots(m+\mu)}.$$

$$\mathrm{B}(m,n)=\int_0^\infty\frac{x^{n-1}}{(1+x)^{m+n}}dx=\frac{m+n}{n}\int_0^\infty\frac{x^n\,dx}{(1+x)^{m+n+1}}$$

$$=\frac{(m+n)(m+n+1)\ldots(m+n+\mu)}{n(n+1)\quad\ldots\quad(n+\mu)}\int_0^\infty\frac{x^{n+\mu}}{(1+x)^{m+n+\mu}}\,d$$

$$=\frac{(m+n)(m+n+1)\ldots(m+n+\mu)}{n(n+1)\ldots(n+\mu)}\int_0^1 x^{n+\mu}(1-x)^{m-1}dx$$

$$=\frac{(m+n)(m+n+1)\ldots(m+n+\mu)}{n(n+1)\ldots(n+\mu)}\cdot\frac{1.2.3\ldots\mu}{m(m+1)\ldots(m+\mu)}\times\frac{\int_0^1 x^{n+\mu}(1-x)^{m-1}dx}{\int_0^1 x^{\mu}(1-x)^{m-1}dx}.$$

If n lies between r and $r+1$

$$\frac{\int_0^1 x^{r+\mu}(1-x)^{m-1}d}{\int_0^1 x^{\mu}(1-x)^{m-1}dx}>\frac{\int_0^1 x^{\mu+n}(1-x)^{m+1}dx}{\int_0^1 x^{\mu}(1-x)^{m-1}dx}$$

$$>\frac{\int_0^1 x^{\mu+r+1}(1-x)^{m-1}dx}{\int_0^1 x^{\mu}(1-x)^{m-1}dx};$$

and

$$\operatorname*{Lt}_{\mu\to\infty}\frac{\int_0^1 x^{r+\mu}(1-x)^{m-1}dx}{\int_0^1 x^{\mu}(1-x)^{m-1}dx}=\operatorname*{Lt}_{\mu\to\infty}\frac{1.2.3\ldots(\mu+r)}{m(m+1)\ldots(m+\mu+r)}$$

$$\times\frac{m\ m+1)\ldots(m+\mu)}{1.2.3\ldots\mu}=\operatorname*{Lt}_{\mu\to\infty}\frac{(\mu+1)\ldots(\mu+r)}{(m+\mu+1)\ldots(m+\mu+r)}=1.$$

A. C. L. Wilkinson.

A Diophantine Problem.

1. If $ax^2-y^2=1$ is a conic passing through the *rational* points (p, q), (p', q'), then we can write

$$\left.\begin{aligned} pa^{\frac{1}{2}}&=\cosh u,\ q=\sinh u\ ;\\ p'a^{\frac{1}{2}}&=\cosh v,\ q'=\sinh v. \end{aligned}\right\}$$

$$\therefore\qquad \left.\begin{aligned} \sinh(u\pm v)&=qp'a^{\frac{1}{2}}\pm pq'a^{\frac{1}{2}},\\ \cosh(u\pm v)&=pp'a\pm qq'. \end{aligned}\right\}$$

Hence, the *rational* points $(qp'\pm pq')$, $(app'\pm qq')$ lie on the conjugate hyperbola $ax^2-y^2=-1$.

The *rational* points may, therefore, be readily constructed as the ends of chords conjugate to the radius to (p, q) or (p', q'); and the process can be continued indefinitely.

Ex. (1). $2x^2-y^2=1$.

Here $(1, 1)$ is an obvious *rational* point. Therefore, if (p, q) is any other such, the points $(q\pm p)$,$(2p\pm q)$ lie on the conjugate $y^2-2x^2=1$; and so on.

Ex. (2). $2x^2-3y^2=1$.

Here, if (p, q) is a point, $\{(5p\pm6q), (4p\pm5q)\}$ is also a point on it.

2. If $ax^2+y^2=1$, we may put $\left.\begin{aligned} pa^{\frac{1}{2}}&=\cos\theta,\ q=\sin\theta,\\ p'a^{\frac{1}{2}}&=\cos\phi,\ q'=\sin\phi, \end{aligned}\right\}$ 0;

and deduce $\left.\begin{aligned} \sin(\theta+\phi)&=(qp'+pq')a^{\frac{1}{2}},\\ \cos(\theta+\phi)&=(app'-qq'.) \end{aligned}\right\}$

Thus $\{(qp'+pq'),(a\,pp'-qq')\}$ is also a point on the ellipse $ax^2+y^2=1$

3. When the equation is $ax^2+by^2=1$, put

$$pa^{\frac{1}{2}}=\cos\theta,\ qb^{\frac{1}{2}}=\sin\theta;$$

and suppose

$$p'=\cos\phi,\ q'\,(ab)^{\frac{1}{2}}=\sin\phi,$$

so that (p', q')lies on

$$x^2+aby^2=1.$$

Then

$$\cos(\theta+\phi)=(pp'-bqq')a^{\frac{1}{2}},$$

$$\sin(\theta+\phi)=(qp'+pq'a)b^{\frac{1}{2}};$$

and the points $\{\pm(qp'+apq'), \pm(pp'-bqq')\}$ will also lie on $ax^2+by^2=1$, being the ends of a chord conjugate to the radius to (p, q).

A similar method applies to $ax^2-by^2=1$.

Example : $2x^2-3y^2=1$.

Take the auxiliary conic $x^2-6y^2=1$, which is satisfied by (5, 2).

Hence if (p, q) is a point on $2x^2-3y^2=1$,

$$\{(5p+6q), (5q-4p)\},$$

also lies on the same, and so on.

M. T. Naraniengar.

Note on Q. 831.

(ENQUIRER) :—Find by elementary methods the invariant relation expressing the condition that hexagons may be inscribed in the conic $S=0$ which are also circumscribed to the conic $S'=0$.

[*N.B.*—The result quoted in Salmon's *Conic Sections*, 6th Edition p. 343, *foot-note*, contains an extraneous factor corresponding to degenerate hexagons. The result is given in Halphen, *Functions Elliptiques*, Tome II.]

Solution by A. O. L. Wilkinson.

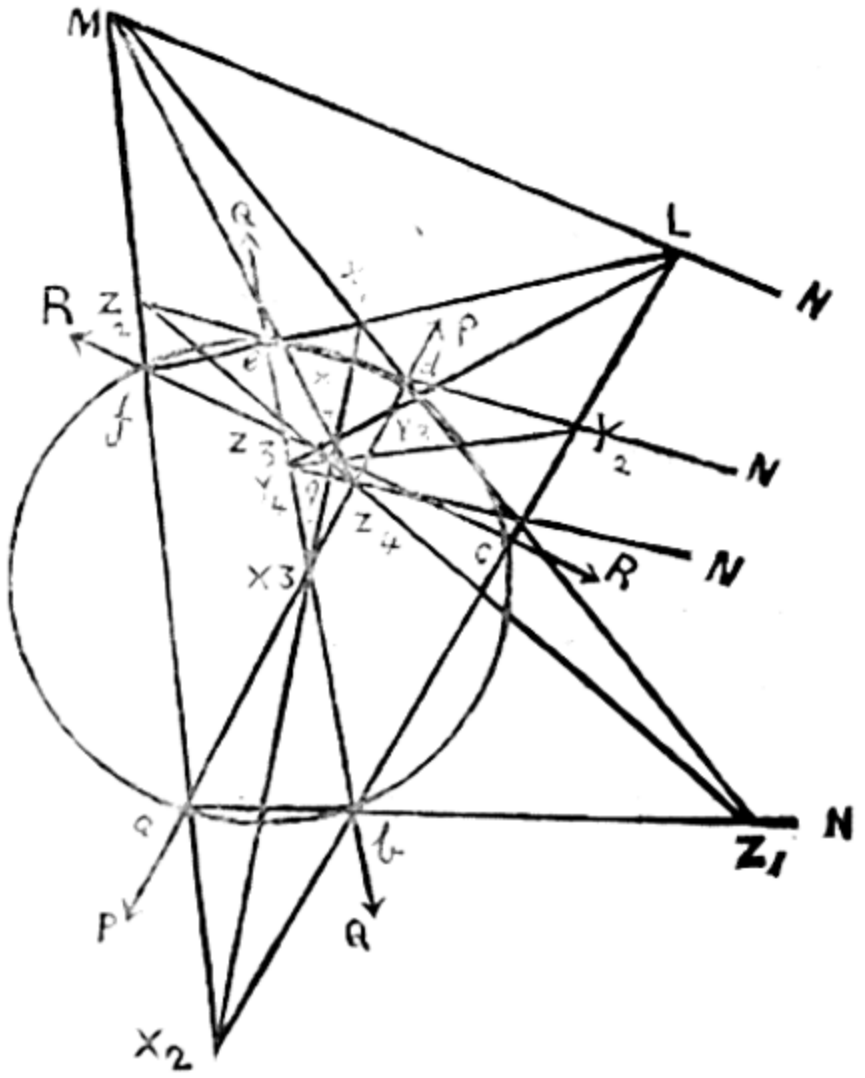


Fig. 1.

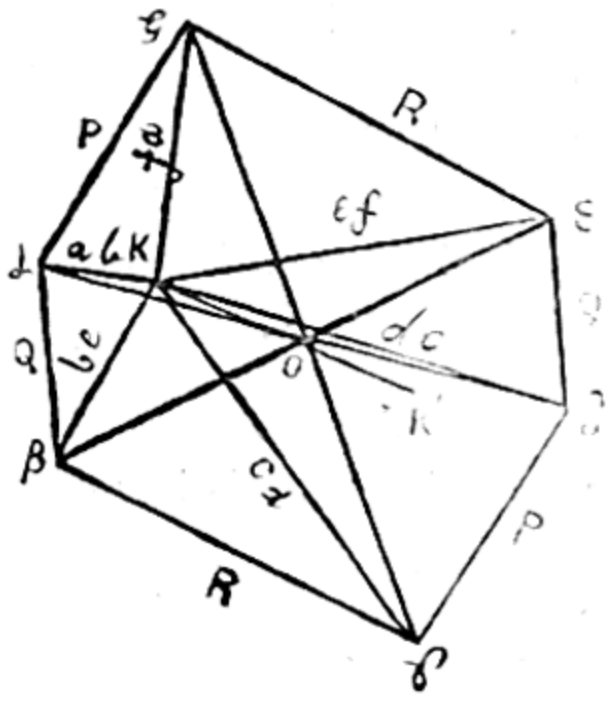


Fig. 2.

§ 1. Consider a jointed framework *abcdef* inscribed in a circle and kept in equilibrium by six equal and opposite forces P, Q, R acting along *ad*, *be*, *cf*. (fig. 1)

The conditions for equilibrium give

$$\frac{P}{\sin a \sin d}=\frac{T_{ab}}{\sin e \sin d}=\frac{T_{af}}{\sin c \sin d}=\frac{Q}{\sin b \sin e}=\frac{T_{bc}}{\sin e \sin f}=\text{etc.}$$

which are symmetrical and thus the frame—work is in equilibrium. Consider the force diagram for any point K (fig. 2) where Kα, Kβ .....are parallel to *ab*, *bc*.....

Then *abcdef* is a funicular corresponding to K.

Now the hexagon $\alpha\beta\gamma\delta\varepsilon\zeta$ is symmetrical about O; thus considering K′, a point such that O is the middle point of KK′, then K′α is parallel to Kδ, etc.

Thus *defabc* is another funicular and the corresponding sides of these two funiculars intersect on a straight line parallel to KK.'

Hence *ab*, *de* ; *bc*, *ef* ; *cd*, *fa* intersect on a straight line NLM, the Pascal line of the hexagon *abcdef*, and this line is parallel to KOK'.

§ 2. Consider any funicular corresponding to O, the centre of the force diagram ; in general it consists of a hexagon inscribed in the triangle $X_3Y_3Z_3$, formed by the lines *ad*, *be*, *cf*, whose sides meet the corresponding sides of the funiculars *abcdef*, *defabc* in points lying on two straight lines parallel to LMN or KOK'.

Consider a straight line NY_4Z_4 parallel to $\alpha O\delta$ and complete the funicular $Z_4Y_4X_4Z_4'Y_4'X_4'$; since corresponding sides of this funicular and *abcdef* intersect on a straight line parallel to OK, that is, parallel to LMN, and L is the point of intersection of Z_4Y_4 and *ab*, therefore LMN is the line of intersection of the corresponding sides. Hence Y_4X_4 passes through L, X_4Z_4' through M and $Z_4'Y_4'$ through N. But $Z_4'Y_4'$ is parallel to $O\delta$ and hence to Y_4Z_4. Thus the funicular degenerates to the triangle $X_4Y_4Z_4$ inscribed in the triangle $X_3Y_3Z_3$, whose sides pass respectively through L, M, N.

§ 3. The pencil at K consisting of $K\beta$, $K\varepsilon$, KO and the line through K parallel to $\beta\varepsilon$ is harmonic ; therefore Lcb, Lef, LMN, LX_4Y_4 form a harmonic pencil.

Produce *ab*, *cd*, *ef* to form the triangle $X_1Y_1Z_1$ and *bc*, *de*, *fa* to form the triangle $X_2Y_2Z_2$.

It follows from the harmonic pencils at L and M that $X_1X_4X_2$ are collinear.

§4. *The points* X_1, X_2, X_3, X_4 *are collinear.*

Consider the forces, P, Q acting at *a*, *b* ; their resultant passes through X_2 for they can be replaced by T_{ba}, T_{fa} and T_{ab}, T_{db} and thus X_3X_2 is the resultant.

Similarly by considering P, Q acting at *d*, *e*, X_3X_1 is the line of action of the resultant.

Again P at Z_4 can be replaced by forces along Z_4Y_4, X_4Z_4 represented by $O\alpha$, αO ; and Q at Y_4 can be replaced by forces along Y_4Z_4 and X_4Y_4 represented by αO, $O\beta$; whence X_4 is also a point on the resultant.

Similarly $Y_1Y_2Y_3Y_4$ and $Z_1Z_2Z_3Z_4$ are collinear.

Let X_3X_4, Y_3Y_4 meet in *g* ; then the moments about *g* for P, Q at *a*, *b* is zero and that about *g* for P, R at *d*, *c* is zero and the P forces are

equal and opposite, hence the sum of the moments about g for Q, R at b, c is zero.

Hence g also lies in Z_3Z_4.

Now $X_1X_2X_3$ is the Pascal line of *afe bcd*, $Y_1Y_2Y_3$ the Pascal line of *abc fed* and $Z_1Z_2Z_3$ of *afcdeb*. Thus g is the Steiner point of the Pascal lines X_1X_2, Y_1Y_2, Z_1Z_2.

The point g is $\begin{pmatrix} af,\ cd,\ be \\ bc,\ fe,\ ad \\ de,\ ab,\ cf \end{pmatrix}$ in the usual notation.

§5. The four triangles $X_1Y_1Z_1$, $X_2Y_2Z_2$, $X_3Y_3Z_3$, $X_4Y_4Z_4$ are copolar with a common pole at g.

The three axes of homology of any three triangles which are copolar with a common pole are known to be concurrent (a very simple proof by projecting the line joining the common pole to the intersection of two of the axes of homology to infinity). The axes of homology are seen to be

(12) the Pascal line LMN, (13)the Pascal line $\begin{pmatrix} ab,\ cd,\ ef \\ cf,\ be,\ ad \end{pmatrix}$

(14) LMN, (23) the Pascal line $\begin{pmatrix} ad,\ cf,\ be \\ bc,\ de,\ af \end{pmatrix}$, (24) LMN and the axis of mology (34).

Hence the 3 Pascal lines

$$\begin{pmatrix} af,\ bc,\ de \\ cd,\ fe,\ ab \end{pmatrix},\quad \begin{pmatrix} af,\ bc,\ de \\ be,\ ad,\ cf \end{pmatrix},\quad \begin{pmatrix} cd,\ fe,\ ab \\ be,\ ad,\ cf \end{pmatrix}$$

are concurrent and the axis of homology of the triangles $X_3Y_3Z_3$, $X_4Y_4Z_4$ also passes through this point of concurrence of the 3 Pascal lines.

Thus the axis of homology of the triangles $X_3Y_3Z_3$, $X_4Y_4Z_4$ meets LMN in a second g point, and we shall show that these two g points are conjugate with respect to the conic. (Standt's Theorem).

§ 6. *The axis of homology of the triangles* $X_3Y_3Z_3$, $X_4Y_4Z_4$ *and the Pascal line LMN are harmonic conjugates with respect to the Pascal lines*

$$\begin{pmatrix} ab,\ cd,\ ef \\ cf,\ be,\ ad \end{pmatrix} \text{ and } \begin{pmatrix} ab,\ cf,\ be \\ bc,\ de,\ af \end{pmatrix}.$$

Project LMN to infinity so that we obtain fig. 3, where *ab*, *de* ; *bc ef* ; *cd*, *af* are respectively parallel. Then $X_4Y_4Z_4$ are the middle points of X_1X_2, Y_1Y_2, Z_1Z_2. Also X_2Z_2, X_4Z_4, X_1Z_1 are three parallel straight lines and hence they meet X_3Z_3 in three points U,V,W where V is the middle point of UW, and this establishes the above theorem. Also since X_4 is

the middle point of *cf*, we see that in fig. 1 the point where *ad* meets LMN is fourth harmonic with respect to Z_1 of *a* and *d*.

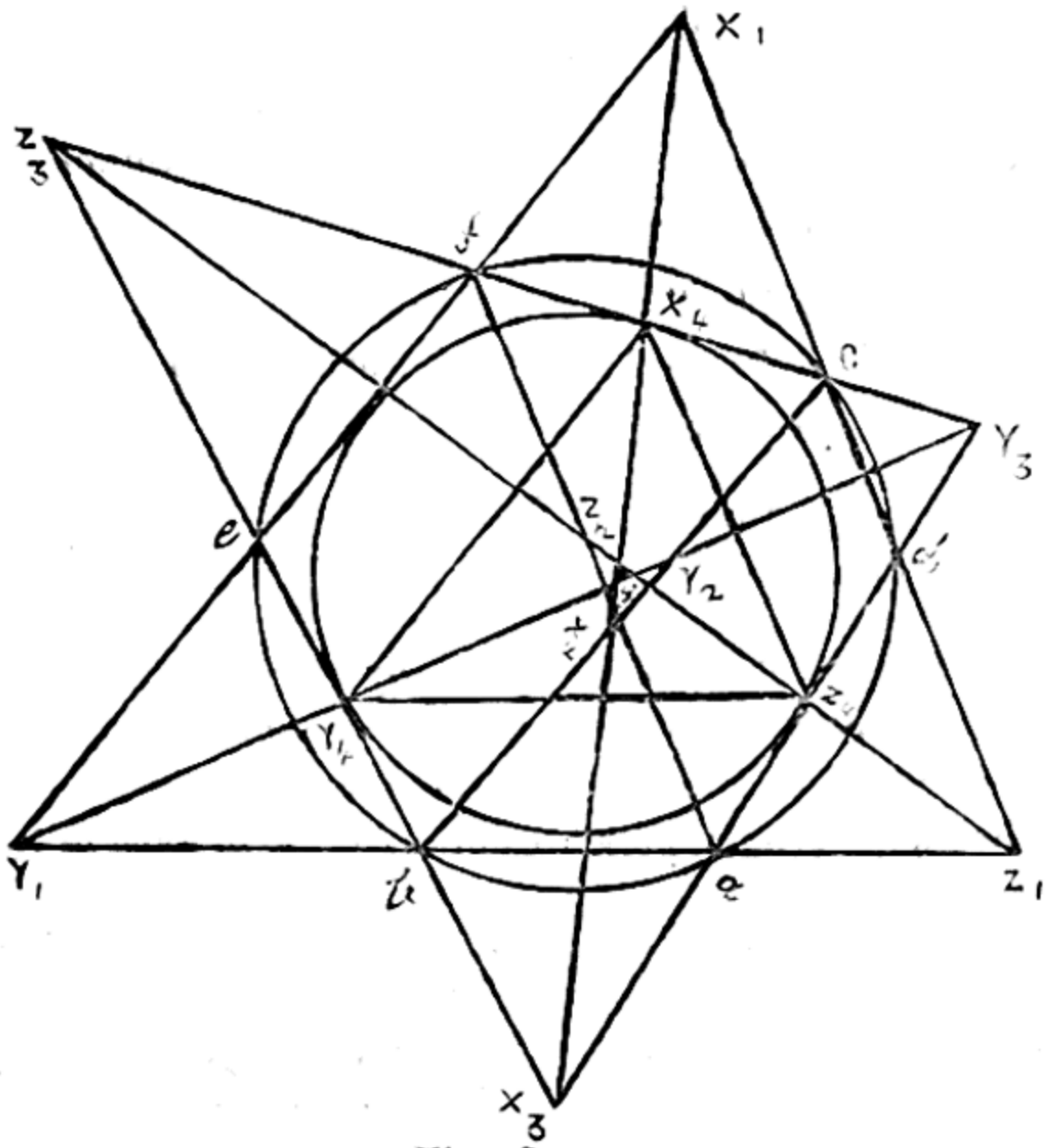


Fig. 3.

§ 7. Taking the conic of fig. 3 as a circle, since $ad=be=cf$ and Z_4, Y_4, X_4 are their middle points, we see that $X_4Y_4Z_4$ lie on a circle inscribed in the triangle $X_3Y_3Z_3$ and concentric with the circle *abcdef*.

Hence in fig. 1. the conics *abcdef* and $X_4Y_4Z_4$ touching the sides of $X_3Y_3Z_3$ have double contact, the Pascal line LMN being the chord of contact; and more generally if we take any other hexagon *a'b'c'd'e'f'* whose verties lie on the lines of action of the forces P, Q, R and whose corresponding sides *a'b'*, *d'e'* pass through N, etc., the conics *a'b'c'd'e'f'* all have double contact at the same points with the Pascal line LMN.

In fig. 3, *g* is the symmedian point of the triangle $X_4Y_4Z_4$ and its polar with respect to the circle $X_4Y_4Z_4$ is the axis of homology of $X_4Y_4Z_4$ and $X_3Y_3Z_3$; hence the polar of *g* with respect to *adcdef* is parallel to the axis of homology of $X_4Y_4Z_4$, $X_3Y_3Z_3$ and thus passes through the point of intersection of LMN (the line at infinity) and the axis of homology of $X_4Y_4Z_4$, $X_3Y_3Z_3$; this was shown in § 5 to be the intersection of the three Pascal lines, *viz.* the reciprocal *g* point. We thus have a proof of Staudt's, Theorem that the two reciprocal *g* points are conjugate with respect to the fundamental conic.

A. C. L. Wilkinson.

Astronomical Notes.

A new minor planet.

Among the recent discoveries of minor planets is one of some importance. This planet (as yet unnamed) has a perihelion distance of 1·182 astronomical units, very little greater than that of Eros, so that on certain favourable occasions it may approach to within about seventeen million miles of the Earth, and will therefore be useful for determining the solar parallax. The eccentricity of the planet's orbit is very great so that its aphelion distance is 3·879. Its period is almost exactly four years. The planet is of the tenth magnitude and its diameter about four miles. The planet was discovered by Wolf who recently announced the discovery of a minor planet attended by a satellite which moved through eight degrees in a little over half an hour—a discovery of so startling a nature that one awaits its confirmation with some curiosity.

In connection with minor planets another point of interest occurs to me. It has been known for many years that at distances from the Sun at which the motion would be commensurate with that of Jupiter, there are no minor planets, any planets which may have originally existed in these positions having been diverted therefrom by the perturbing action of Jupiter. A recent investigation of the orbits of the many hundreds of planets now known fully confirms these facts, gaps being found at the ratios 2/1, 7/3, 5/2, 8/3, 3/1. It is however found that for values of the daily motion less than 500″ this avoidance of commensurability does not hold and planets are found with ratios 1/1, 4/3, 3/2.

The planets with ratio 1/1 are a special case, forming with Jupiter and the Sun an equilateral triangle and illustrating Lagrange's particular solution of the problem of three bodies; there are four such planets known, usually referred to as the Trojan group since they are named after the heroes of the Trojan war· Hector, Patroclus, &c.

R. J. Pocock.

SOLUTIONS.

Question 64.

(A. C. L. WILKINSON, M.A.):—Prove that no solution in positive integers exists of the equations:

$$x^2-y^2=y^2-z^2=z^2-u^2.$$

Solution by H. Br.

1. *Lemma.* The equations

$$\alpha^2=\beta^2+\delta^2=\gamma^2+9\delta^2 \quad \ldots \quad (1)$$

have no solution in integers, if $\delta \neq 0$.

Proof: (i) A common factor f of any two of the quantities is a factor of the other two also. [This is easily seen except possibly in the case where $f=2$ and is a factor of β, γ. In this case

$$\left(\frac{\beta}{2}\right)^2-\left(\frac{\gamma}{2}\right)^2=2\delta^2.$$

$\therefore$ $\frac{\beta}{2}, \frac{\gamma}{2}$ are both odd or both even; hence the difference of their squares is a multiple of 4; *i.e.*, δ is even].

We may therefore without loss of generality take α, β, γ, δ prime to each other. As the sum of two odd squares is not a square δ is even.

(ii) By the ordinary rules, we must have

$$m^2+n^2=\alpha=M^2+N^2 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

$$2mn=\delta\ ;\ 2\,M\,N=3\,\delta$$

or

$$3\,m\,n=MN \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

where m n are co-primes and n is even; and M, N are co-primes and N is even.

From (3) we get

either $\left.\begin{matrix} m=a\,b, & M=3\,a\,c \\ n=c\,d & N=b\,d \end{matrix}\right\}$ $\ldots \quad \ldots \quad \ldots \quad (4)$

or $\left.\begin{matrix} m=a\,b & M=a\,c \\ n=c\,d & N=3\,b\,d \end{matrix}\right\}$ $\ldots \quad \ldots \quad \ldots \quad (5)$

a, b, c, d being prime to each other, and d being even. We take the two cases separately.

(iii) From (2) and (4),

$$a^2\,b^2+c^2\,d^2=9\,a^2\,c^2+b^2\,d^2$$

or

$$b^2\,(d^2-a^2)=c^2\,(d^2-9a^2)$$

giving

$$\frac{d^2-a^2}{c^2}=\frac{d^2-9a^2}{b^2}=\text{an integer}=v.$$

$\therefore$ $8\,a^2$ is divisible by c^2-b^2 which must therefore be the common factor of a^2 and d^2.

As a, b, c, d are prime to each other and b and c are odd, v can only be $+1$ or -1.

If $v=1$, $d^2=a^2+c^2$, which is impossible as d is even and, a c are odd.

If $v=-1$, $9a^2=b^2+d^2$ which is also impossible as b, d are not both multiples of 3.

Hence the relations (4) do not hold.

(iv) From (2) and (5) we get

$$a^2 b^2+c^2 d^2=a^2 c^2+9b^2 d^2$$

or
$$\frac{a^2-d^2}{b^2}=\frac{a^2-9d^2}{c^2}=\pm 1.$$

The lower sign gives $9d^2=a^2+c^2$ which is impossible as a, c are not both multiples of 3.

The upper sign gives

$$a^2=b^2+d^2=c^2+9d^2 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (6)$$

an equation of the same form as (1). Also $a=\frac{m}{b}<m^2+n^2<\alpha$.

(v) Proceeding in this manner we get a series of equations with diminishing values of α, and ultimately arrive at the equations

$$1=b_o^2+d_o^2=c_o^2+9d_o^2$$

which have no solution (if $d_o \neq 0$).

2. To prove that the system of equations

$$x^2-y^2=y^2-z^2=z^2-u^2$$

has no integral solution except the trivial one

$$x^2=y^2=z^2=u^2.$$

(i) Clearly x, y, z, u may be taken to be prime to each other without loss of generality $x^2=2y^2-z^2$

We may solve this by putting

$$x+z=X,\ x-z=Y,$$

or we may apply known results in the theory of quadratic forms, and noting that

$$x^2=2y^2-z^2=(2y+z)^2-2(y+z)^2$$

we may at once put down

$$\left.\begin{aligned} \pm x &= A^2-2B^2 \\ 2y+z &= A^2+2B^2 \\ y+z &= 2AB \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (7)$$

where A is odd, and A and B are prime to each other.

Similarly

$$\left.\begin{aligned} \pm u &= C^2-2D^2 \\ y+2z &= C^2+2D^2 \\ y+z &= 2CD \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (8)$$

C being odd, and C and D prime to each other.

Eliminating y and z

$$A^2+C^2+2B^2+2D^2=6AB=6CD \quad \ldots \quad \ldots \quad \ldots \quad (9)$$

As AB=CD, we put

$$\left.\begin{aligned} A=pr,\ B=qs \\ C=ps,\ D=qr \end{aligned}\right\} \quad \dots \quad \dots \quad \dots \quad \dots \quad (10)$$

Then p, q, r, s are prime to each other and p, r, s are odd

(ii) Substituting in (9)

$$p^2r^2+p^2s^2+2q^2s^2+2q^2r^2=6pqrs$$

or
$$(p^2+2q^2)(r^2+s^2)=6pqrs$$

Now r^2+s^2 has no factor common with r, s and p^2+2q^2 has no factors common with p, q. Also r^2+s^2 is even, but is not a multiple of 3

$\therefore$
$$\left.\begin{aligned} p^2+2q^2=3rs \\ 2pq=r^2+s^2 \end{aligned}\right\}$$

or
$$\left.\begin{aligned} 2(p+q)(2q+p)=3(r+s)^2 \\ 2(p-q)(2q-p)=3(r-s)^2 \end{aligned}\right\} \quad \dots \quad \dots \quad (11)$$

It follows that $r+s$ may be put in the form $2\alpha\beta$, where α, β are prime to each other, and $r-s$ may be put in the form $2\gamma\delta$ where γ, δ are prime to each other. As $p^2-q^2=-3r^2+3rs=$ a multiple of 3, and as p and q are not both multiples of 3, one of these quantities $p+q$, $p-q$ is a multiple of 3 but not the other. Two cases arise

(1)
$$\left.\begin{aligned} p+q=6\alpha^2, \\ 2p+q=\beta^2 \\ 2q-p=3\gamma^2 \\ p-q=2\delta^2. \end{aligned}\right\} \quad \dots \quad \dots \quad \dots \quad (12)$$

Eliminating p and q, we get

$$9\alpha^2=\beta^2+\delta^2,$$

or β, δ are multiples of 3, implying r, s multiples of 3, which is not the case as r, s are co-prime's. $\therefore$ (12) is inadmissible.

(2) In the second case

$$\left.\begin{aligned} p+q=2\alpha^2 \\ 2q+p=3\beta^2 \\ 2q-p=\gamma^2 \\ p-q=6\delta^2. \end{aligned}\right\} \quad \dots \quad \dots \quad \dots \quad \dots \quad (13)$$

Eliminating pq, we get

$$\left.\begin{aligned} \alpha^2=\beta^2+\delta^2 \\ =\gamma^2+9\delta^2 \end{aligned}\right\}$$

This has no integral solutions unless $\delta=0$

If $\delta=0$, $r=s$

$$A=C,\ B=D$$
$$y=z,$$

and finally

$$x^2=y^2=z^2=u^2.$$

———

Question 378.

(J. C. SWAMINARAYAN) :—Having given that

$$\{\Sigma(b+c)x^2-2\Sigma(fyz)\}^2=4(x^2+y^2+z^2)[\Sigma(bc-f^2)x^2+2\Sigma(gh-af)yz],$$

prove that $\dfrac{bz^2+cy^2-2fyz}{y^2+z^2}=\dfrac{cx^2+az^2-2gzx}{z^2+x^2}=\dfrac{ay^2+bx^2-2hxy}{x^2+y^2}$

Remarks by H. Br.

The given expression may be written

$$[(a+b+c)(x^2+y^2+z^2)-(ax^2+by^2+cz^2+2fyz+2gzx+2hxy)]^2$$
$$=4(x^2+y^2+z^2)(Ax^2+By^2+Cz^2+2Fyz+2Gzx+2Hxy).$$

Take (x, y, z) to be co-ordinates of a point, the axes being supposed to be rectangular. As $a+b+c$ is an invariant of the quadric $ax^2+\ldots$ it is clear that the above expression retains its form if the axes are rotated in any way. Consequently, the conclusions (whatever they be) to be drawn from the data will also retain their form unaltered if the axes are rotated in any way. But the conclusion that is asserted in the question does not satisfy the test.

To verify: Suppose the quadric referred to its principal axes, so that $f=g=h=0$.

$$[(b+c)x^2+(c+a)y^2+(a+b)z^2]^2=4(x^2+y^2+z^2)(bcx^2+cay^2+abz^2).$$

Expanding and simplyfying we get

$$\Sigma(b-c)^2x^4-2\Sigma(a-b)(c-a)y^2z^2=0.$$

Which implies

$$\sqrt{(b-c)}.x\pm\sqrt{(c-a)}.y\pm\sqrt{(a-b)}.z=0. \quad \ldots \quad \ldots \quad (1)$$

We are required to prove that

$$\frac{bz^2+cy^2}{y^2+z^2}=\frac{cx^2+az^2}{x^2+z^2}=\frac{ay^2+bx^2}{x^2+y^2}$$

i.e.

$$\left.\begin{aligned} z^2[(b-c)x^2+(c-a)y^2-(a-b)z^2]&=0\\ y^2[(b-c)x^2-(c-a)y^2+(a-b)z^2]&=0\\ x^2[-(b-c)x^2+(c-a)y^2+a-b)z^2]&=0 \end{aligned}\right\} \quad \ldots \quad \ldots \quad (2)$$

The set of relations (2) is not a necessary consequence of (1).

Question 723.

(S. RAMANUJAN) :—If $[x]$ denote the greatest integer in x and n is a positive integer, show that

(i) $\left[\frac{n}{3}\right]+\left[\frac{n+2}{6}\right]+\left[\frac{n+4}{6}\right]=\left[\frac{n}{2}\right]+\left[\frac{n+3}{6}\right]$,

(ii) $[\frac{1}{2}+\sqrt{(n+\frac{1}{2})}]=[\frac{1}{2}+\sqrt{(n+\frac{1}{4})}]$,

(iii) $[\sqrt{n}+\sqrt{(n+1)}]=[\sqrt{(4n+2)}]$,

Solution by H. Br.

(i) Put $\phi(n)=\left[\frac{n}{3}\right]+\left[\frac{n+2}{6}\right]+\left[\frac{n+4}{6}\right]-\left[\frac{n}{2}\right]-\left[\frac{n+3}{6}\right]$ ()

4

Let $\left[\frac{n}{6}\right]=k$, so that $n=6k+m$, where $m<6$.

$$\left[\frac{n}{3}\right]=\left[2k+\frac{m}{3}\right]=2k+\left[\frac{m}{3}\right], \text{ \&c.}$$

$\therefore$ $\phi(n)=\phi(m)$, where $m=0, 1, 2, 3, 4,$ or 5.

It is easily verified that $\phi(m)=0$ and therefore also $\phi(n)=0$.

(ii) Put $n=k^2+a$, where $a<2k+1$. Then $[\sqrt{n}\,]=k$.

Let
$$n+\tfrac{1}{2}=k^2+a+\tfrac{1}{2}=(k+\zeta)^2,$$
$$n+\tfrac{1}{4}=k^2+a+\tfrac{1}{4}=(k+\eta)^2.$$

We are required to prove $[\frac{1}{2}+\sqrt{n+\frac{1}{2}}]=[\frac{1}{2}+\sqrt{n+\frac{1}{4}}]$

or, in effect,
$$[\tfrac{1}{2}+\xi]=[\tfrac{1}{2}+\eta]$$

If $a<k$, $0<\eta<\xi<\frac{1}{2}$, and $[\frac{1}{2}+\xi]=[\frac{1}{2}+\eta]=0$,

If $a\geq k$, $\frac{1}{2}\leq\eta<\xi<1$, and $[\frac{1}{2}+\xi]=[\frac{1}{2}+\eta]=1$.

(iii) As before let $n=k^2+a$, where $a<2k+1$.

Put

$$\left.\begin{aligned} n&=k^2+a=(k+\xi)^2,\\ n+\tfrac{1}{2}&=k^2+a+\tfrac{1}{2}=(k+\eta)^2,\\ n+1&=k^2+a+1=(k+\zeta)^2. \end{aligned}\right\} \quad \ldots \quad \ldots \quad (1)$$

We are required to prove

$$[\sqrt{n}+\sqrt{n+1}]=[\sqrt{4n+2}]$$

or in effect
$$[\xi+\zeta]=[2\eta].$$

If $a<k$, we have $0\leq\xi<\eta<\zeta<\frac{1}{2}$, and $[\xi+\zeta]=[2\eta]=0$

If $a>k$, we have $\frac{1}{2}<\xi<\eta<\zeta\leq 1$, and $[\xi+\zeta]=[2\eta]=1$.

If $a=k$, $\xi<\frac{1}{2}$, and η, ζ are $>\frac{1}{2}$, but <1. Thus $[2\eta]=1$. We have to show that $[\xi+\zeta]=1$. If $\xi=\frac{1}{2}-x$, and $\zeta=\frac{1}{2}+z$, we have to show that $z>x$.

Substituting for ξ and ζ in (1), we get
$$\tfrac{3}{4}=z\,(2k+1)+z^2,$$
and
$$-\tfrac{1}{4}=-x\,(2k+1)+x^2$$
$$\therefore \quad (z-x)(2k+1)=\tfrac{1}{2}-(x^2+z^2).$$

As x and z are both less than $\frac{1}{2}$, x^2+z^2 is $<2(\frac{1}{2})^2<\frac{1}{2}$.

$\therefore$ $(z-x)(2k+1)$ is positive, or $z-x$ is positive.

Hence in every case $[\xi+\zeta]=[2\eta]$.

Question 836.

(Martyn M. Thomas, M.A.):—At a given instant, the same star is observed to be at the horizon of one place (lat. ϕ_1, long. L_1); on the prime vertical of another place (lat. ϕ_2, long. L_2) and at the zenith of a third place whose latitude is unknown, and long. L_3. Prove that

$$\cos(L_3-L_2)\tan\phi_1+\cos(L_3-L_1)\cot\phi_2=0,$$

and find the latitude of the third place.

Solution by R. J. Pocock, S Muthukrishnan, K. R. Rama Iyer, A. K. Anantanarayanan, R. D. Karve and C. Krishnamachary.

Let the hour angles of the star be h_1 h_2 in the first two cases; then

$$h_1 = L_s - L_1 ; \quad h_2 = L_s - L_2 ;$$

and if δ is the star's declination $\delta = \phi_s$.

Also
$$\tan \delta . \tan \phi_1 = -\cos h_1,$$
$$\tan \delta . \cot \phi_2 = \cos h_2.$$

Eliminating δ, h_1, h_2, we have

$$\cos (L_s - L_1) \cot \phi_1 + \cos (L_s - L_2) \tan \phi_2 = 0,$$

or
$$\cos (L_s - L_2) \tan \phi_1 + \cos (L_s - L_1) \cot \phi_2 = 0 ;$$

and
$$\tan \phi_s = \tan \delta = \cos (L_s - L_1) \tan \phi_1$$
$$= -\cos (L_s - L_1) \cot \phi_2.$$

Question 843.

(Lakshmishanker N. Bhatt):—From any point C tangents CD and CE are drawn to a circle, and from any fixed point O on the circle, chords OD, OE are drawn. If from any other variable point O′ on the circle, chords O′D, O′E are drawn cutting the former chords in P, Q and the tangents in R, S; then prove that (i) the line PQ always passes through C and (ii) the line RS always touches a conic.

Solution (1) *by Hemraj*; (2) *by V. M. Gaitonde, S. R. Ranganathan, and C. Bhaskaraiya, B.A. (Hon.)*

(1) Reciprocating with respect to O, we have :—

TP, TQ are tangents to a parabola at P, Q; and a variable tangent cuts them in R, S. Show that the lines through R and S parallel to TQ and TP respectively meet on PQ, and (ii) the locus of the point of intersection of the diagonals PS, RQ of the quadrilateral PQSR is a conic.

Take TP, TQ as axes. The parabola is

$$\sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}} = 1,$$

and a variable tangent is $\frac{x}{f} + \frac{y}{g} = 1$ with the condition $\frac{f}{a} + \frac{g}{b} = 1$, ... (1)

(i) $$PQ \equiv \frac{x}{a} + \frac{y}{b} - 1 = 0$$

and the point of intersection of lines through R, S parallel to TQ, TP is evidently (f, g). It lies on PQ, if

$\frac{f}{a}+\frac{g}{b}-1=0$, which is (1)

[For a geometrical proof, see Lachlan's *Modern Pure Geometry* Art 272].

(ii) $$PS\equiv\frac{x}{a}+\frac{y}{g}-1=0 \quad \ldots \quad \ldots \quad (2)$$

$$QR\equiv\frac{x}{f}+\frac{y}{b}-1=0. \quad \ldots \quad \ldots \quad (3)$$

Eliminating f, g between (1), (2), (3) we have the locus required, *viz.*

$$x^2/a^2+xy/ab+y^2/b^2-2x/a-2y/b+1=0.$$

It has double contact with the parabola at P and Q. Hence the theorem.

(2) (i) Let OO′ and DE meet in K. Then by the quadrangle construction for a polar, PQ is the polar of K. Also since K lies on DE the polar of C, the polar of K always passes through C.

Thus PQ always passes through C.

(ii) It can easily be shown that R and S generate homographic ranges on CE, CD respectively and that the ranges are not in perspective. Therefore RS touches a conic touching the fixed lines CD, CE.

Additional Solutions by S. V. Venkatachala Iyer and H. R. Kapadia.

Question 846.

(S. Malhari Rao, B.A) :—Solve in positive integers $10^x+95^x=101^y$

Solution (1) *by H. Br.* (2) *by S. V. Venkatachala Aiyar and N. B. Mitra.*

(1) $$10^x+95^x\equiv 0 \pmod{101}$$

Multiply both sides by $(-17)^x$, and simplify. We get

$$32^x+1\equiv 0$$

or $$2^{5x}+1\equiv 0 \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

As 2 is not a quadratic residue of 101, $2^{50}+1\equiv 0$.

Henceif x_o is the lowest positive value of x satisfying (1), 50 must be an odd multiple of 5 x_o.

$$\therefore \qquad x_o=2 \text{ or } 10.$$

$x_o=2$, gives $2^{5x}\equiv 2^{10}\equiv 1024 \pm -1$; $x_o=10$ is obviously admissible.

The general value of x is any odd multiple of x_0 ;

or $$x=20k+10 ;$$

the value of y can be deduced when x is known.

(2) Forming the residues to the modulus 101 of the successive powers of 10^x, we have

$$10, -1, -10, 1, \ldots\ldots$$

(the same set recurring after this),

Similarly the residues of 95^x are

$$-6,\ 36,-14,-17,+1,\ldots\ldots$$

(the same set recurring).

In order that 10^x+95^x may be divisible by 101, the sum of their residues must be zero. This is obviously the case when x has any of the values 10, 30, 50,..... ; for, when x has a value of the form $2+4n$, the residue in the first case is -1, and when x has a value of the form $5m$, the residue in the second case is $+1$;

Thus the values of x having been found, the corresponding values of y can be easily calculated.

Question 861.

(Communicated by Mr. J. H. Grace, F. R. S.):—A heterogeneous medium is such that the path of every ray through it is a plane curve. Find the index of refraction at any point.

Solution by Balak Ram.

If μ be the index of refraction at any point, the differential equations of a ray are

$$\frac{d}{ds}\left(\mu\frac{dx}{ds}\right)=\frac{\partial\mu}{\partial x},$$

and two similar equations. Denoting $\frac{d\theta}{ds}$ by θ', we have

$$\mu x''=\frac{\partial\mu}{\partial x}-x'\mu',$$

and

$$\mu x'''=-2\mu'x''-\frac{\partial\mu'}{\partial x}-x'\mu'', \text{ \&c.}$$

If the path is a plane curve, the torsion is zero. Simplifying the determinant factor of the value of the torsion, we get

$$\begin{vmatrix} x' & y' & z' \\ \frac{\partial\mu}{\partial x} & \frac{\partial\mu}{\partial y} & \frac{\partial\mu}{\partial z} \\ \frac{\partial\mu'}{\partial x} & \frac{\partial\mu'}{\partial y} & \frac{\partial\mu'}{\partial z} \end{vmatrix}=0.$$

We must have relations of the form

$$\left.\begin{aligned} \frac{\partial\mu'}{\partial x}+u\frac{\partial\mu}{\partial x}&=vx',\\ \frac{\partial\mu'}{\partial y}+u\frac{\partial\mu}{\partial y}&=vy',\\ \frac{\partial\mu'}{\partial z}+u\frac{\partial\mu}{\partial z}&=vz'. \end{aligned}\right\}$$

The first equation is equivalent to

$$d\left(e^{\int u ds}\frac{\mu}{\partial\mu}\right)=ve^{\int u ds}dx.$$

This shows that the right hand side is a perfect differential.

Therefore $ve^{\int u ds}$ is independent of y and z. Similarly by considering the other equations, we find that it is independent of x, and is consequently a constant. Integrating, we get

$$\frac{\partial\mu}{\partial x}=v(x-x_0),$$

$$\frac{\partial\mu}{\partial y}=v(y-y_0),$$

$$\frac{\partial\mu}{\partial z}=v(z-z_0);$$

or $$d\mu=v\,[(x-x_0)dx+(y-y_0)dy+(z-z_0)dz].$$

Changing to polar co-ordinates, the new origin being taken to be at (x_0, y_0, z_0),

$$d\mu\equiv\frac{\partial\mu}{\partial r}dr+\frac{\partial\mu}{\partial\theta}d\theta+\frac{\partial\mu}{\partial\phi}d\phi=vrdr.$$

$$\therefore\qquad\frac{\partial\mu}{\partial\theta}=\frac{\partial\mu}{\partial\phi}=0,$$

or $$\mu=f(r)=F[(x-x_0)^2+(y-y_0)^2+(z-z_0)^2],$$

where (x_0, y_0, z_0) is an arbitrary point and F is any function.

Question 874.

(S. Malhari Rao) :—Solve in primes

$$x+y=206,\ x+z=132,\ z+w=82.$$

Solution by H. Br. N. B. Mitra, C. Krishnamachary, K. B. Madhava, R. D. Karve, R. J. Pocock, G. Joshi and S. V. Venkatachala Aiyer.

As x, y, z, w are primes, and the value 2 is clearly inadmissible for any of them, each of them is of the form $6k\pm1$. From the given relations if follows that x and y are of the form $6k+1$, and z and w of the form $6k-1$. The values of w that make w and $82-w$ prime are

$$w=11,\ 23,\ 29,\ 41,\ 53,\ 59,\ 71,$$

$$z=82-w=71,\ 59,\ 53,\ 41,\ 29,\ 23,\ 11\ ;$$

of these $w=29, 53, 59$ give prime values of x and y, as follows :—

x	y	z	w
79,	127,	53,	29
103,	103,	29,	53
109,	97,	23,	59

Question 875.

(M. T. NARANIENGAR) :—P is any point on the circle of similitude of two circles A, B; and Q, R are the inverses of P with respect to A, B. Show that QR is bisected by the radical axis of A and B.

Solution (1) *by Hemraj, S. Muthukrishnan, and K. R. Rama Iyer,*

(2) *by Hemraj, M. K. Kewalramani, and S. V. Venkatachala Iyer.*

(1) Denote the centres of A and B by A and B, and the circle of similitude by C.

Since P is on C, $\frac{AP}{BP}=\frac{e}{e_1}$, where e, e_1 are the radii of A, B;

and since Q,R are the inverses of P *w.r.t.* A, B,

$$\therefore \quad \frac{AP}{e}=\frac{e}{AQ}, \text{ and } \frac{BP}{e_1}=\frac{e_1}{BR}$$

$$\therefore \quad \frac{e}{AQ}=\frac{e_1}{BR} \text{ i.e. } \frac{AP}{BP}=\frac{AQ}{BR}$$

QR ∥ AB *i.e.* the radical axis of A, B is ⊥ QR. ... (1)

Evidently the circle PQR is orthogonal to A, B and C which is coaxal with A, B,

Thus the centre of the circle PQR is on the radical axis and on the right bisector of QR as well. Hence by (1) the radical axis bisects at right angles QR unless it is parallel to QR, *i.e.*, unless the centre of the circle PQR is at infinity, *i.e.*, unless P coincides with either centre of similitude; even in this case QR is bisected by the radical axis.

(2) Taking the line of centres and the radical axis of A, B as axes we have the equation of A, B in the form

$$x^2+y^2-2kx+\delta=0 \quad x^2+y^2-2k_1x+\delta=0.$$

Hence the equation of C is

$$x^2+y^2-2\frac{kk_1+\delta}{k+k_1}x+\delta=0. \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

Now P (x_1, y_1) is any point on (1),

$$\therefore \quad x_1^2+y_1^2-2\frac{kk_1+\delta}{k+k_1}x_1+\delta=0; \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

and Q is the point of intersection of the polar of P *w.r.t.* A and the line through the centre of A perpendicular to the polar; *i.e.* the intersection of the lines

$$(x_1-k)x+y_1y-kx_1+\delta=0, \; y_1x-(x_1-k)\,y-y_1k=0.$$

Hence Q is $\left(\frac{k_1-k}{k+k_1-2x_1}x_1, \frac{k+k_1}{k+k_1-2x_1}y_1\right)$. ... ...

Similarly R is $\left(\frac{k-k_1}{k+k_1-2x_1}x_1, \frac{k+k_1}{k+k_1-2x_1}y_1\right)$.

Hence the esult follows at once.

Question 878.

(S. Narayana Aiyar, M.A.):—Show that

$$(1-k^2x^2)^m(1-x^2)^n=\sum_{r=0}^{\infty}\frac{(-x^2)^r}{\Gamma(r+1)}\cdot\frac{\Gamma(n+1)}{\Gamma(n-r+1)}F(-m,-r,n-r+1,k^2)$$

when F denotes the hypergeometric function.

Hence show that

$$\int(1-k^2x^2)^{-\frac{1}{2}}(1-x^2)^{-\frac{1}{2}}dx=\sum_{r=0}^{\infty}\frac{(-)^rx^{2r+1}}{\Gamma(r+1)(2r+1)}\cdot\frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}-r)}\times F(\tfrac{1}{2},-r,\tfrac{1}{2}-r,k^2).$$

Solution by K. B. Madhava, S. R. Ranganathan, C. Krishnamachary and Sadanand.

If neither m nor n is a positive integer,

$$(1-k^2x^2)^m=\sum_{r=0}^{\infty}\frac{\Gamma(m+1)}{\Gamma(m-r+1)\,\Gamma(r+1)}k^{2r}(-x^2)^r$$

and

$$(1-x^2)^n=\sum_{r=0}^{\infty}\frac{\Gamma(n+1)}{\Gamma(n-r+1)\,\Gamma(r+1)}(-x^2)^r.$$

Hence the coefficient of $(-x^2)^r$ in the product $(1-x^2)^n(1-k^2x^2)^m$ is

$$\sum_{s=0}^{s=r}\frac{\Gamma(n+1)\,\Gamma(m+1)}{\Gamma(n-r+1)\,\Gamma(r+1)}\cdot k^{2s}\frac{\Gamma(n-r+1)\,\Gamma(r+1)}{\Gamma(m-s+1)\Gamma(r-s+1)\Gamma(s+1)\Gamma(n-r+s+1)}$$

$$=\frac{\Gamma(n+1)}{\Gamma(n-r+1)\Gamma(r+1)}\sum_{s=0}^{s=r}\frac{\Gamma(m+1)}{\Gamma(m-s+1}\cdot\frac{\Gamma(r+1)}{\Gamma(r-s+1)}\cdot\frac{\Gamma(n-r+1)}{\Gamma(n-r+s+1)}\cdot\frac{k^{2s}}{\Gamma(s+1)}$$

$$=\frac{\Gamma(n+1)}{\Gamma(n-r+1)\,\Gamma(r+1)}F(-m,-r,n-r+1,k^2),$$

where the hypergeometric series contains only a finite, *viz.* $r+1$ terms.

Since neither m nor n is a positive integer, the product can be summed for r fron 0 to ∞.

Hence $$(1-k^2x^2)^m(1-x^2)^n=\sum_{r=0}^{\infty}\frac{(-x^2)^r}{\Gamma(r+1)}\cdot\frac{\Gamma(n+1)}{\Gamma(n-r+1)}F(-m,-r,n-r+1,k^2).$$

The second result follows by a simple integration, and putting $m=-\frac{1}{2}$ and $n=-\frac{1}{2}$.

Question 889.

(HEMRAJ) :—Two parabolas have each double contact with a given ellipse of semi-axes a, b, and their axes parallel to its axes respectively PP′, QQ′, are the chords of contact, and ρ_1, ρ_2 the radii of curvature of the parabolas; and ρ, ρ', those of the ellipse at P, Q. If the distances of the chords of contact from the centre of the ellipse are in the ratio of its axes, prove that $\rho\,\rho_2=\rho'\rho_1$. In particular if the distances are equal, then

$$a^2\,\rho\,\rho_2=b^2\,\rho'\,\rho_1.$$

But if Q coincides with P, then

$$\frac{1}{\rho}=\frac{1}{\rho_1}+\frac{1}{\rho_2}.$$

Solution (1) *by K. Appukuttan Erady and K. B. Madhava;*

(2) *by C. Krishnamachary, S. V. Venkatachala Aiyer and Sadanund.*

(1) Let the equations to the chords P P′ and Q Q′ be $x-f=0$ and $y-g=0$ respectively.

Then the equations to the two parabolas are

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1-\frac{1}{a^2}(x-f)^2=0, \quad \ldots \quad \ldots \quad \text{(i)}$$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1-\frac{1}{b^2}(y-g)^2=0. \quad \ldots \quad \ldots \quad \text{(ii)}$$

For the ellipse at P,

$$\frac{y^2}{b^2}=1-\frac{x^2}{a^2},$$

$$\frac{y}{b^2}\cdot\frac{dy}{dx}=-\frac{x}{a^2},$$

$$\frac{y}{b^2}\cdot\frac{d^2y}{dx^2}=-\frac{1}{a^2}-\frac{1}{b^2}\left(\frac{dy}{dx}\right)^2.$$

For the parabola (i) at P,

$$\frac{y^2}{b^2}=1-\frac{2fx}{a^2}+\frac{b^2}{a^2},$$

$$\frac{y}{b^2}\cdot\frac{dy}{dx}=-\frac{f}{a^2},$$

$$\frac{y}{b^2}\cdot\frac{d^2y}{dx^2}=-\frac{1}{a^2}-\frac{1}{b^2}\left(\frac{dy}{dx}\right)^2.$$

Since the parabola (i) touches the ellipse at P, their curvatures are in the ratio of their $\frac{d^2y}{dx^2}$ at P.

Hence in the notation of the Question

$$\frac{\rho}{\rho_1}=\frac{\frac{1}{b^2}\left(\frac{dy}{dx}\right)^2}{\frac{1}{a^2}+\frac{1}{b^2}\left(\frac{dy}{dx}\right)^2}=\frac{\frac{f^2b^2}{a^4y^2}}{\frac{1}{a^2}+\frac{f^2b^2}{a^4y^2}}=\frac{\frac{b^2}{a^2}}{\frac{y^2}{b^2}+\frac{f^2}{a^2}}=\frac{f^2}{a^2}.$$

And it follows from symmetry that at R,

$$\frac{\rho'}{\rho_2}=\frac{g^2}{b^2}.$$

Hence if $\frac{f}{a}=\frac{g}{b}$,

$$\frac{\rho}{\rho_1}=\frac{\rho'}{\rho_2},$$

or

$$\rho\rho_2=\rho'\rho_1;$$

and if $f=g$,

$$\frac{a^2\rho}{\rho_1}=\frac{b^2\rho'}{\rho_2},$$

or

$$a^2\rho\rho_2=b^2\rho'\rho_1.$$

Also if P coincides with Q,

$$\rho'=\rho \text{ and } \frac{f^2}{a^2}+\frac{g^2}{b^2}=1,$$

and consequently

$$\frac{\rho}{\rho_1}+\frac{\rho}{\rho_2}=1,$$

i.e.,

$$\frac{1}{\rho}=\frac{1}{\rho_1}+\frac{1}{\rho_2}.$$

(2) Let the eqnation of the ellipse be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

and P, Q, be the points whose excentric angles are α, β.

If P′ be α', the eqn. to the parabola with its axis parallel to the x—axis is

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)-k\left\{\frac{x}{a}\cos\frac{\alpha+\alpha'}{2}+\frac{y}{b}\sin\frac{\alpha+\alpha'}{2}-\cos\frac{\alpha-\alpha'}{2}\right\}=0.$$

Since the axis of this parabola is parallel to the x-axis,

$$k=1, \text{ and } \alpha+\alpha'=0;$$

and the eqnation to the parabola is

$$\frac{y^2}{b^2}+\frac{2x}{a}\cos\alpha-(1+\cos^2\alpha)=0. \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

Similarly the equation to the parabola with Q Q′ as chord of contact and axis parallel to the y—axis is

$$\frac{x^2}{a^2}+\frac{2y}{b}\sin\alpha-(1+\sin^2\alpha)=0. \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

Now $$\rho=\frac{(a^2\sin^2\alpha+b^2\cos^2\alpha)^{\frac{3}{2}}}{ab}, \quad \rho'=\frac{(a^2\sin^2\beta+b^2\cos^2\beta)^{\frac{3}{2}}}{ab}.$$

Also we can easily find ρ_1, ρ_2, the radii of curvature of the parabolas (1) and (2) at P, Q, and these are given by

$$\rho_1=\frac{(a^2\sin^2\alpha+b^2\cos^2\alpha)^{\frac{3}{2}}}{ab\cos^2\alpha},\ \rho_2=\frac{(a^2\sin^2\beta+b^2\cos^2\beta)^{\frac{3}{2}}}{ab\sin^2\beta}.$$

The perpendiculars from the origin on the chords of contact PP' QQ' are $a\cos\alpha$, $b\sin\beta$.

If these are in the ratio of the axes

$$\cos\alpha=\sin\beta\ ;\ \rho\rho_2=\rho'\rho_1.$$

If these are equal, $a\cos\alpha=b\sin\beta$,

$$a^2\rho\rho_2=b^2\rho'\rho_1.$$

But if P, Q coincide, $\alpha=\beta$, and the relation

$$\frac{1}{\rho}=\frac{1}{\rho_1}+\frac{1}{\rho_2}$$

easily follows.

Question 898.

(HEMRAJ):—A parabola has double contact with a given ellipse of semi-axes a, b and its axis parallel to either axis of the latter. Find the equation of the locus of the centre of curvature of the parabola corresponding to the point of contact; and show that

$a^2\,b^2\,\rho_1^2\,\rho_2^2=\{(\rho_2-\rho_1)\,a^2+\rho_1\,b^2\}^2$, or $\{(\rho_2-\rho_1)\,b^2+\rho_1\,a^2\}^2$

where ρ_1, ρ_2 are the radii of curvature of the curves at a point of contact.

Solution by K. B. Madhava and C. Krishnamachary.

We notice that

$$y^2=4\,A\,(x+C) \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

is a parabola with its axis parallel to the major axis of

$$1=\frac{x^2}{a^2}+\frac{y^2}{b^2} \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

and having double contact with it, if

$$4\,AC\,b^2=b^4+4\,A^2\,a^2. \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

The points of contact of (1) and (2) are

$$-\frac{2\,A\,a^2}{b^2},\ \pm\frac{1}{b}\sqrt{b^4-4\,A^2\,a^2}$$

where ρ_1^2 (Ellipse) $=\frac{1}{a^2b^2}(a^2-x^2\,e^2)=\frac{a^4}{b^{14}}(b^4-4\,A^2a^2\,e^2)^3$ ... (4)

and ρ_2^2 (Parabola) $=\frac{4}{A}(A+x+C)^3=\frac{1}{16\,A^4\,b^6}(b^4-4A^2\,a^2\,e^2)^3$; (5)

whence the required relation is got by eliminating A between (4) and (5) *i.e.* between (4), and $\rho_1 b^4 = 4 A^2 a^2 \rho_2$.

$$\therefore \qquad \rho_1^2 = \frac{a^4 b^{12}}{b^{14} \rho_2^5} (\rho_2 - \rho_1 e^2)^3,$$

i.e. $$a^2 b^2 \rho_1^2 \rho_2^3 = \{ a^2 (\rho_2 - \rho_1) + b^2 \rho_1 \}^3.$$

Similarly the other result (when the axis of the parabola is parallel to the minor axis of the ellipse) can be got by interchanging a and b.

To obtain the locus of the centre of curvature, we eliminate x, y, A and C from (1), (3) and

$$\xi = 2A + 2C + 3x \text{ and } \eta = -\frac{y^3}{4A^2}.$$

Question 904.

(R. J. Pocock, B.A., B. Sc.) :—Show that

$$\int_0^{\frac{\pi}{2}} \frac{\sin^{2n}\phi}{\left(1-\frac{1}{2}\sin^2\phi\right)^{n+\frac{1}{2}}} d\phi = 2^{n-1} \frac{\Gamma^2\left(\frac{2n+1}{4}\right)}{\Gamma\left(\frac{2n+1}{2}\right)}.$$

Solution (1) *by F. H. V. Gulasekharam and by the Proposer*; (2) *by A. O. L. Wilkinson and K. R. Rama Iyer.*

(1) Consider $$I = \int_0^\infty x^{2n} e^{-x^4} dx = \frac{1}{4}\Gamma\left(\frac{2n+1}{4}\right);$$

by substituting $x^4 = y$

whence $$2^{2n} I^2 = 2^{2n} \int_0^\infty \int_0^\infty x^{2n} y^{2n} e^{-x^4 - y^4} dx dy$$

$$= \int_0^\infty \int_0^{\frac{\pi}{2}} r^{4n} \sin^{2n} 2\theta \, e^{-r^4 (1 - \frac{1}{2}\sin^2 2\theta)} r \, dr \, d\theta$$

$$= \frac{1}{4} \int_0^\infty \int_0^\pi z^{2n} \sin^{2n}\phi \, e^{-z^2 (1 - \frac{1}{2}\sin^2\phi)} \, dz \, d\phi$$

$$= \frac{1}{2} \int_0^\infty \int_0^{\frac{\pi}{2}} \frac{\sin^{2n}\phi}{\left(1-\frac{1}{2}\sin^2\phi\right)^{\frac{2n+1}{2}}} \cdot t^{2n} e^{-t^2} dt \, d\phi$$

$$= \frac{1}{4} \Gamma\left(n+\frac{1}{2}\right) \int_0^{\frac{\pi}{2}} \frac{\sin^{2n}\phi}{\left(1-\frac{1}{2}\sin^2\phi\right)^{\frac{2n+1}{2}}} d\phi$$

This solution is suggested by a similar analysis, due to Forsyth given in G. H. Darwin's paper "On the Mechanical Conditions of a Swarm of Metorites" (Phil. Trans. Series A, Vol. 180 (1880) pp. 1—69; also Collected Papers, Vol. IV, pp. 362—431).

Putting $n=0$, we get $\Gamma\left(\frac{1}{4}\right)=2\pi^{\frac{1}{4}}\mathrm{F}^{\frac{1}{2}}$,

where F is the complete elliptic integral of the first kind, modulus $\frac{1}{\sqrt{2}}$. From $\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)=\pi\sqrt{2}$, and Legendres formula connecting the complete elliptic integrals of the first and second kind, which for modulus $\frac{1}{\sqrt{2}}$ becomes $2\,\mathrm{FE}-\mathrm{F}^2=\frac{1}{2}\pi$, we obtain

$$\Gamma\left(\frac{3}{4}\right)=\pi^{\frac{1}{4}}(2\,\mathrm{E}-\mathrm{F})^{\frac{1}{2}};$$

and we thus obtain the expressions for

$$\int_0^\infty e^{-x^4}dx,\quad \int_0^\infty x^2 e^{-x^4}dx$$

given in the paper quoted above; and more generally

$$\int_0^\infty x^{2n}e^{-x^4}dx=\frac{1}{4}\Gamma\left(\frac{2n+1}{4}\right)$$

can be reduced to one or other of these two expressions.

(2) Write $\sin x=\dfrac{\frac{1}{2}\sin^2\phi}{1-\frac{1}{2}\sin^2\phi}$, $\cos x=\dfrac{\cos\phi}{1-\frac{1}{2}\sin^2\phi}$; so that

$$\cos x\,dx=\frac{\sin\phi\cos\phi d\phi}{\left(1-\frac{1}{2}\sin^2\phi\right)^2},\quad dx=\sqrt{2\sin x}\,\frac{d\phi}{\left(1-\frac{1}{2}\sin^2\phi\right)^{\frac{1}{2}}}$$

$$\therefore\qquad \mathrm{I}=2^{n-\frac{1}{2}}\int_0^{\frac{1}{2}\pi}\sin x^{n-\frac{1}{2}}\,dx=2^{n-1}\,\frac{\Gamma^2\left(\frac{2n+1}{4}\right)}{\Gamma\left(\frac{2n+1}{2}\right)}.$$

Question 905.

(T. P. BHASKARA SASTRI) :—Investigate the effect of precession and nutation on the position angle of a double star.

[See : Ball's *Spherical Astro* : p. 185, Ex. 16.]

Solution by C. Krishnamachary and the Proposer.

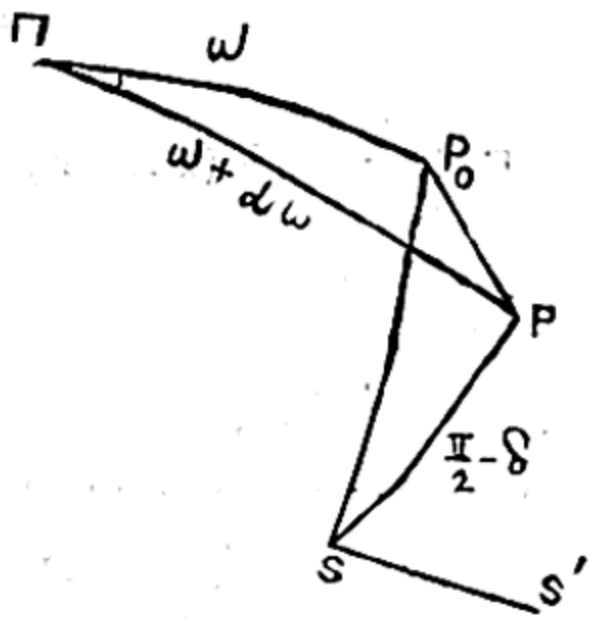
Let π be the pole of the ecliptic.

P_o, the celestial pole at epoch T_o.

P, the same at a subsequent epoch T.

S, the principal star and S′ his companion

Fig. 1.



also let the position angle at the first epoch be p_o, and p that at the second epoch.

$\angle$ P π P_o = the arc between the poles of the circles πP_o and π P
$= k$.

From the triangle P_o S P,

$$\cos\delta \sin(p-p_o) = -\sin P P_o \sin \angle P P_o S.$$

But $\angle P P_o S = \angle P P_o \pi - \angle S P_o \pi = \angle P P_o \pi - (90° + \alpha_o)$
$= -(90° - \angle P P_o \pi) - \alpha_o$.

$$\therefore \quad \cos\delta \sin(p-p_o) = \sin P P_o \sin P P_o \pi \sin \alpha_o + \sin P P_o \cos P P_o \pi \cos \alpha_o$$

From the triangle P P_o π,

$\sin P P_o \sin P P_o \pi = \sin(\omega + d\omega) \sin k$ and $\sin P P_o \cos P P_o \pi = \cos(\omega + d\omega) \sin\omega - \sin(\omega + d\omega) \cos\omega \cos k$.

Making these substitutions, we get

$$(p-p_o)\cos\delta = [k \sin\omega \sin\alpha - d\omega \cos\alpha],$$

neglecting small quantities of order higher than the first.

i.e. $$p - p_o = [n \sin\alpha - l\omega \cos\alpha] \sec\delta.$$

QUESTIONS FOR SOLUTION.

903. *Correction :* Insert at the end of the question the following :—" where the summation is extended only to odd values of n ".

960. (V. Ramaswami Aiyar, M.A.) :—Given a triangle ABC, if a point P and the feet of the perpendiculars from it on the sides and altitudes of ABC all lie on a Cartesian Oval, show the negative pedal of the oval with respect to P touches the nine points circle. Show that the pedals of the inscribed and escribed circles of a triangle, taken with respect to the orthocentre, are cases of such ovals, illustrating the property.

961. (H. Br.) :—Show that

$$\left.\begin{aligned} x^2 &= y^2 + z^2 \\ t^2 &= y^2 + 4z^2 \end{aligned}\right\}$$

have no solutions in non-zero integers.

962. (A. C. L. Wilkinson) :—If two circles are such that hexagons can be inscribed in the one which are circumscribed to the other, prove that the sum of the products of the diagonals taken two at a time is constant and equal to

$$\frac{4\,R^2 r^2 \{ (R^2 - c^2)^2 + 4\,R^2 r^2 \}}{(R^2 - c^2)^2},$$

where R, r, are the radii of the two circles and c the distance between their centres.

963. (A. C. L. Wilkinson) :—If two circles are such that quadrilaterals can be inscribed in the one which are circumscribed to the other, the ratio of the areas of these quadrilaterals is constant.

964. (H. Br.) :—If $f(x, y)$ is a rational algebraic function of x and y, such that $f(-x, y) = (-)^{k+1} f(x, y)$,

(i) Show that

$$\int_0^\infty f(\sin 2\theta, \cos 2\theta) \frac{d\theta}{\theta^{k+1}}$$

$$= \frac{(-1)^k}{k!} \int_0^\infty f(\sin 2\theta, \cos 2\theta) \tan \theta . \frac{d^k}{d\theta^k} \cot \theta . \frac{d\theta}{\theta}$$

$$= \frac{(-1)^k}{k!} \int_0^{\frac{\pi}{2}} f(\sin 2\theta, \cos 2\theta). \frac{d^k}{d\theta^k} \cot \theta. \, d\theta$$

and

$$\text{(ii)} \int_0^{\infty} f(\sin\theta, \cos\theta) \frac{d\theta}{\theta^{k+1}} = \frac{(-1)^k}{k!} \int_0^{\frac{\pi}{2}} f(\sin\theta, \cos\theta) \frac{d^k\lambda}{d\theta^k} \, d\theta$$

where $\lambda = \cot\theta$ or $\operatorname{cosec}\theta$, according as

$$f(-x,-y) = f(x, y) \text{ or } -f(x, y).$$

[The integrals are assumed to be finite.]

965. (HEMRAJ) :—ABC is a triangle. A point J is taken on AI_1 produced, such that I_1J subtends at B or C an angle α where $\cot\alpha = 3\cot\frac{1}{2}C$ or $3\cot\frac{1}{2}B$. Show that there exists a circle through A and J which cuts AB and AC in D and E such that $BD = CE = BC$.

[Suggested by Q. 959.]

965. (A. C. L. WILKINSON) :—Prove that

$$\frac{1-2(2+k^2)s_1^2+5k^2s_1^4}{2s_1c_1d_1} + \frac{1-2(1+k^2)s_2^2+3k^2s_2^4}{2s_2c_2d_2} - \frac{1-2(1+2k^2)s_3^2+5k^2s_3^4}{2s_3c_3d_3}$$
$$= -3k^2s_1s_2s_3$$

where $s_1 = \operatorname{sn}\left(\frac{K}{3}\right) s_2 = \operatorname{sn}\frac{iK'}{3}$, $s_3 = \operatorname{sn}\frac{K+iK'}{3}$ etc.

966 (A. C. L. WILKINSON) :—Prove that

$$'E - 3E\left(\frac{K}{3}\right) = -\frac{k^2}{k'} \operatorname{cn}\frac{K}{3}.$$

967. (M. K. KEWALRAMANI, M.A) :—Prove that

$$1^{-5}+2^{-5}-4^{-5}-5^{-5}+7^{-5}+8^{-5}-10^{-5}-11^{-5}+\ldots\ldots = \frac{35\pi^6}{8748\sqrt{3}}.$$

968 (V. ANANTARAMAN) :—Sum the series

$$\frac{1}{3}-\frac{1}{7}+\frac{1}{11}-\frac{1}{15}+\frac{1}{19}-+\ldots\ldots$$

List of Periodicals Received.

(*From 16th March to* 15*th May* 1918).

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1. American Mathematical Monthly, Vol. 24, Nos 9 & 10, Nov. & Dec. 1917.
2. Annuals of Mathematics, Vol. 19 No. 2, December 1917.
3. Astrophysical Journal, Vol 46 Nos. 4 and 5, November and December 1917.
4. Bulletin of the American Mathematical Society, Vol. 24. Nos. 4, 5 & 6 January, February and March 1918. (2 Copies).
5. L'Intermediaire der Mathematiciens, Vol 25, Nos 1 and 2, Jan and Feb. '18
6. Mathematical Gazette, Vol 9 No. 133, January 1918. (3 Copies).
7. Mathematics Teacher, Vol 10, No. 2, December 1917.
8. Messenger of Mathematics, Vol 47, Nos 4, 6 & 7, Agust, Oct. & Nov. 1917.
9. Monthly Notices of the Royal Astronomical Society, Vol 78, Nos. 1 & 2, November and December 1917.
10. Philosophical Magazine, Vol 35, Nos. 206 and 207, Feb. and March 1918.
11. Popular Astronomy Vols 25 and 26, Nos 10 and 1 and 2; December 1917 January and February 1918. (3 Copies).
12. Proceedings of the Royal Society of London, Vol 94, No. 659,
13. School Science and Mathematics, Vol. 18, Nos. 1 and 2, Jan. and Feb. 1918. (2 Copies).
14. Tohoku Mathematical Journal, Vol 13, Nos 1 and 2, February 1918.
15. Transactions of the American Mathematical Society, Vol 19, No. 1, Jan. '18,

Properties of Equations in the Algebra of the Nodal Cubic.

By R. Vythynathaswamy.

[The present paper purposes to give a formal development of some general function—properties in the Algebra of the Nodal Cubic, on the basis of the results already established from geonretrical considerations in my previous paper on the subject.

The following is a general summary of the paper.

A. Discussion of Functional Forms.

(1) The general functional form transformable into a successive product *i.e.* a product in which the multiplier at each stage is a single element instead of being itself a product. The successive product taken as the universal standard form.

(2) Classification of forms into pure and mixed forms with respect to a variable x. Reduction of the general mixed form to three canonical types. Consequent reduction of the general equation

$$F_1(x) = F_2(x)$$

to two fundamental types,

$$P_x = a \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

$$P_x = a\,(\text{or } Q_x) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

where P_x, Q_x are pure functions.

(3) Classification of forms into two types, the simplex and the complex. A form $f(x)$ is complex if $f(n)$ is indeterminate and simplex if $f(n)=n$ or n^1 (n,n^1 being the nodal elements) A complex form transformable into a simplex one by the rules of the Algebra.

The conditions for the simplexity of the form

$$x^{p_1}(nP_{n-1} - - P_2P_1)$$

are (i) the secondary indices $p_2\ p_3 - - p_n$ are alternately even and odd or odd and even and (ii) $p_1\ p_2$ are both odd or both even.

B. Function—Properties.

(1) The apparent degree of a function when in a complex form higher than its true degree; *e. g.* $x^2 x^2$, a complex form of the function xs

The degree of a simplex form, in other words the true degree of any function is never a multiple of three.

(2) Classification of functions into odd and even functions. $f(x)$ is odd if $f(n)=n^1$ and even, if $f(n)=n$. The odd-ness or even-ness of $f(x)$ directly determinable if the form $f(x)$ is simplex. Whether the form $f(x)$ is simplex or complex. $f(x)$ is odd or even according as the characteristic is positive or negative.

(3) Classification of equations in complex and simplex types according as they are or are not satisfied by the nodal points. The degree of a simplex equation of the first type never a multiple of 3, the degree of a simplex equation of the second type always a multiple of 3.

(4) The problem of solving the general equation reduced to the root-extraction of a given correspondence (which is merely the problem of root extraction of a number and the geometrical problem of drawing a ray making with three rays a pencil of given cross-ratio.

The n roots of an equation of the n th degree form a periodic group of the homographic correspondence of period n whose fixed points are the nodal points.]

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§ 1. *Reciprocal points*

Two points on the Nodal Cubic connected by the relation $ab^2=a^2$s will be said to be reciprocal. This relation is of the third degree in a and there are therefore three a's for any b. One of these three roots is however b itself. Thus there are two points reciprocal to a given point b (not including b itself). We may verify that these two pointe are $\omega_1(\omega_2 b)$ and $\omega_2(\omega_1 b)$ where $\omega_1, \omega_2, \omega_3$ are the points of inflexion. It may be noted that $\omega_1(\omega_2 x)=\omega_2(\omega_3 x)=\omega_3(\omega_1 x)$. This follows from the theorem $\omega_1(\omega_2\omega_3 x)=\omega_2 x$, proved in para (*b*), 9 '*Nodal Cubic*,' J. I. M. S. August 17.

For

$$b(\omega_1(\omega_2 b))^2=b(\omega_1(\omega_2 b^2))=b^2(\omega_1(\omega_2 b)).$$

Thus the two points reciprocal to b are $\omega_1(\omega_2 b)$ and $\omega_2(\omega_1 b)$.

We notice also that these two points are themselves reciprocal.

For

$$\begin{aligned}\omega_1(\omega_2(\omega_1(\omega_2 b)))&=\omega_1(b(\omega_1\omega_2))\\&=\omega_1(\omega_3 b)\\&=\omega_2(\omega_1 b)\end{aligned}$$

which proves $\omega_1(\omega_2 b)$ is reciprocal to $\omega_2(\omega_1 b)$.

Note (1). If Ω denote the homographic correspondence having the nodal points for its fixed points and carrying ω_1 to ω_2 (so that $\Omega(x) = \omega_1(\omega_2 x)$), then Ω carries ω_2 to ω_3 and ω_3 to ω_1 Ω is therefore periodic, of period three. The two points reciprocal to b are then seen to be $\Omega(b)$ and $\Omega^2(b)$.

Note (2). The only self-reciprocal points are the nodal points.

Note (3). If x_1, x_2, x_3 form a mutually reciprocal set,

$$x_1^2 = x_2 x_3,\ x_2^2 = x_3 x_1,\ x_3^2 = x_1 x_2.$$

For
$$x_2 x_3 = [\omega_1(\omega_2 x_1)][\omega_1(\omega_3 x_1)]$$
$$= \omega_1(\omega_1 x_1^2) = x_1^2.$$

Note (4). The points ω_1, ω_2, ω_3 form a mutually reciprocal set, so also do the points, $-\omega_1, -\omega_2, -\omega_3$, which are the points of contact of tangents from the inflexional points.

Note (5). $x = \Omega(x^2).\Omega^2(x)$ identically.

For $\Omega(x^2) = \omega_1(\omega_2 x^2)$ and $\Omega^2(x) = \omega_1(\omega_3 x)$

$\therefore$ $\Omega(x^2).\Omega^2 x = \omega_1^2(\omega_2\omega_3.(x.x^2))$ (by the theorem $ab.cd = ac.bd$).

$$= \omega_1(\omega_1(x.x^2) = x$$

Note (6). $[\Omega(x)]^{2n} = \Omega(x^2)$

From *Note* (5) $\Omega(x^2) = x.\Omega^2(x)$

$$= [\Omega(x)]^2 (\textit{Note}\ (3)).$$

By successive applications of this, the general result follows.

§ 2. The general equation of the third degree may be reduced to one of the two forms

$$x^2 = bx \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1)$$
$$ax^2 = bx \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

b is evidently one root of equation (1). Further if x is one nodal point n, x^2 is the other nodal point n^1 and bx is also the other nodal point n^1, so that the two other roots of equation (1) are n, n^1 the two nodal points. Equation (1) is thus degenerate and need not be considered further.

Consider now eqnation (2). It can be written in the form

$$x^2 = \mathrm{S}\ (x), \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

S denoting a homographic correspondence with the nodal points as fixed points.

The three roots of equation (2) *form a mutually reciprocal set.*

To prove this we shew that if x_1 is one root $\Omega(x_1)$ is another. We have

$$\begin{aligned}[\Omega(x_1)]^2 &= \Omega(x_1^2) \{ \text{I, Note (6).} \} \\ &= \Omega\, S(x_1) \{ \therefore x_1 \text{ is a root} \} \\ &= S\, \Omega(x_1);\end{aligned}$$

so that $\Omega(x_1)$ is another root. Thus the three roots of equation (2) are of the form x_1, $\Omega(x_1)$, $\Omega^2(x_1)$.

The equation (2) thus associates each homographic correspondence with nodal points as fixed points, with a reciprocal set. The identical correspondence is associated with the inflexional triad, and the fundamental involution with the triad $-\omega_1, -\omega_2, -\omega_3$.

§ 3. *Solution of the cubic equation.*

Let Ω_x denote a homographic correspondence, with the nodal points as fixed points, carrying x to a point of inflexion. Thus Ω_x is not definite, but may be any one of the correspondences, $\begin{vmatrix} x \\ \omega_1 \end{vmatrix}$, $\begin{vmatrix} x & x \\ \omega_2 & \omega_3 \end{vmatrix}$. But it is easy to see that in any case, Ω_{x^3} is definite and unique and that if Ω_x^3 is known, x is determined as one of the set of mutually reciprocal points.

$$(\Omega_x^3)^{-3}(\omega_1),\ \Omega(\Omega_x^3)^{-1}(\omega_1)\ \Omega^2(\Omega_x^3)^{-3}(\omega_1).$$

We have the important relation

$$\underline{\Omega_x^3 = \begin{vmatrix} x \\ x^2 \end{vmatrix}}$$

For
$$\begin{aligned}\Omega_x^3(x) &= x(\omega(x(\omega(x(\omega x))))) \\ &= x^2.\end{aligned}$$

This property leads immediately to the solution of the equation

$$x^2 = S(x).$$

For since $x^2 = \Omega_x^3(x)$ identically and Ω_x and S have the same fixed points, it follows that

$$\Omega_x^3 = S = \text{a known correspondence}$$

whence x is determined as shewn above and the equation is solved.

§ 4. *Properties of successive products.*

Consider the successive product.

$$x_n(x_{n-1}(x_{n-2}(\ldots\ldots\ldots\ldots x_2\, x_1)))$$

We shall find it convenient to write this

$$(x_n\, x_{n-1}\ldots\ldots\ldots x_2\, x_1)$$

Let us call $x_1\ x_3\ x_5$...the odd elements and $x_2\ x_4$ ...the even elements of the product. The following results readily follow from (7) para (6) 'Nodal cubic'.

(1) All the even elements and the first odd element can be arranged among themselves in any order.

(2) All the odd elements except the first can be arranged among themselves in any order.

Theorem I. If $P_m = (p_m\ p_{m-1} \cdots p_2\ p_1)$ and $X_m = (p_1\ p_m\ p_{m-1} \cdots p_2 x)$, then $P_m X_m = x$ or $p_1^2\ x$, according as m is odd or even.

For $P_m X_m = (p_m\ p_{m-1} \cdots p_2\ p_1) \times (p_1\ p_m\ p_{m-1} \cdots p_2\ x)$
$= (p_1\ p_{m-1}\ p_{m-2} \cdots p_2\ p_1) \times (p_m\ p_m\ p_{m-1} \cdots p_2\ x)$
$= (p_1\ p_{m-1} \cdots p_2\ p_1) \times (p_{m-1}\ p_{m-2} \cdots p_2\ x)$
$= (p_{m-1}\ p_{m-1}\ p_{m-2} \cdots p_2\ p_1) \times p_1\ p_{m-2} \cdots p_2\ x)$
$= (p_{m-2}\ p_{m-3} \cdots p_2 p_1) \times (p_1 p_{m-2} \cdots p_2 x)$
$= P_{m-2} \times X_{m-2}$
$= P_2 \times X_3$ if m is odd
$= (p_2 p_2 p_1) \times (p_1 p_3 p_2 x)$
$= p_2 \times p_2 x = x$;
and $= P_2 \times X_2$ if m is even
$= p_2 p_1 \times (p_1 p_2 x)$
$= p_1^2 x$.

Theorem II. The product of two successive products can be expressed as a successive product.

Let the two successive products be denoted by P_m, Q_n where,

$$P_m = (p_m p_{m-1} \cdots p_2 p_1),\ Q_n = (q_n q_{n-1} \cdots q_2 q_1).$$

We may suppose m to be odd. If it were even, we may write p_1 as $p_1 p_1^2$ and thus convert P_m into an odd product. We shew that

$$P_m . Q_n = R \text{ where } R = (p_1 p_m \cdots p_2 q_n \cdots q_1).$$

We have

$P_m R = P_m (p_1 p_m \cdots p_2 q_n q_{n-1} \cdots q_1)$
$= P_m (p_1 p_m \cdots p_2 Q_n)$
$= P_m (P_m Q_n)$ (Th I)
$= Q_n$

$\therefore$ $P_m Q_n = R =$ a successive product.

Corollary. It follows by repeated application of them thereto that the most general function of a variable x, can be expressed as a successive product the elements of which are powers of x and constants.

Def. A successive product the elements of which are powers of x is said to be *a pure product.* If constants also appear, the product is said to be *mixed.*

Theorem III The most general mixed product can be reduced to one of the forms, $a\,P_x$, $a\,(bP_x)$, $x^{2m}\,(cP_x)$, where P_x is a pure product. consider first the mixed product C_x, where

$$C_x = x_1(c_1(x_2(c_2 \ldots x_n(c_n X))))$$

$x_1, x_2 \ldots X$ being pure functions of x and c's constants.

$$\begin{aligned}
\text{This product} &= (x_1 X x_2 c_2 x_3 c_3 \ldots x_n c_n') \\
&\quad \text{where } c_n' = c_1 c_n \\
&= (x_1 X x_2 c_2 c_n' c_3 x_4 c_4 \ldots x_{n-1} c_{n-1} x_n') \\
&\quad \text{where } x_n' = x_n x_3 \\
&= (x_1 X x_2 b x_4 c_4 \ldots x_{n-1} c_{n-1} x_n') \\
&\quad \text{where } c_2(c_n'(c_3 d)) = bd \\
&= (x_1 X C_x')
\end{aligned}$$

where C_x' is of the same form as C_x but with fewer elements

Repeating this process of reduction, we would have finally

$$C_x = (y_1 y_2 y_3 \ldots y_n b y)$$

or

$$C_x = (y_1 y_2 y_3 \ldots y_n a b y)$$

where the y's are pure functions of x.

In the second case, C_x at once reduces to the form $a(bP_x)$ where P_x is a pure product. In the first case, C_x would reduce to one of the two forms bP_x, or $y_1\,(bP_x)$.

The latter form can be reduced to $x^{2m}(bP_x)$ by interchanging y_1 and an adequate power of x occuring in P_x expressed as a successive product.

Consider now the general mixed product. The constants occuring in it can be all shifted to the left-hand side of the product, so that if x_r, x_s be the first non-constant odd and even elements, no odd element to the right of x_r and no even element to the right of x_s are constants. The product then reduces to one of the forms C_x, $a\,C_x$, $a(bC_x)$. Taking each of the three possible forms of C_x, it is easy to verify that the product must reduce to one of the three forms mentioned in the theorem.

§ 5. Let the correspondence $\begin{vmatrix} nn^1x \\ nn^1x^2 \end{vmatrix}$ or as we shall write it for convenience $\begin{vmatrix} x \\ x^2 \end{vmatrix}$, be denoted by S_x. Then $\begin{vmatrix} x^2 \\ x^4 \end{vmatrix}$ would be denoted by Sx^2

$$\text{Now } S_x{}^3S_{x^2}(y) = x(x^2(x(x^2x(x^2(x^4y)))$$
$$= x(x^3(x(x^4y)))$$
$$= x(y(x(x^4x^3)))$$
$$= y.$$
$$\therefore \quad S_x{}^2S_{x^2} = 1 \text{ So that } S_{x^2} = S_x{}^{-2}.$$
$$\therefore \left| \begin{matrix} x \\ x^2{}^{n} \end{matrix} \right| = \left| \begin{matrix} x \\ x^3 \end{matrix} \right| \cdot \left| \begin{matrix} x^2 \\ x^4 \end{matrix} \right| \cdots \left| \begin{matrix} x^2{}^{n-1} \\ x^2{}^{n} \end{matrix} \right|$$
$$= S_x . S_x{}^3 \ldots S_x{}^{2^{n-1}}$$
$$= S_x \frac{1-(-2)^n}{3}$$
$$\therefore \left| \begin{matrix} x^2{}^{m_1} \\ x^2{}^{m_2} \end{matrix} \right| = S_x \frac{(-2)^{m_1}-(-2)^{m_2}}{3}$$

Writing the successive product $x^{2^{m_n}}(x^{2^{m_{n-1}}} \ldots (x^{2^{m_2}} x^{2^{m_1}}))$ as $x^{2^{(m_n\, m_{n-1} \cdots m_1)}}$, we may shew more generally

$$\frac{1+(-)^n\{(-2)^{m_1}+(-2)^{m_2}-(-2)^{m_3}+-\}}{3}$$

that $$L \equiv \left| \begin{matrix} x \\ x^{2^{(m_n\, m_{n-1}\, m_1)}} \end{matrix} \right| = S_x$$

Proof. Assume the theorem to be true for products of n elements. We shall prove it true for products of $(n+1)$ elements. We have

$$\left| \begin{matrix} x \\ x^{2^{(m_{n+1},\, m_n \cdots m_1)}} \end{matrix} \right| = L \times \left| \begin{matrix} x^{2^{(m_n\, m_{n-1} \cdots m_1)}} \\ x^{2^{(m_{n+1}\, m_n \cdots m_1)}} \end{matrix} \right|$$
$$= L \times \left| \begin{matrix} x^{2^{m_{n+1}}} \\ x_{2^{(m_n' m_{n-1}' \cdots m_2' m_1')}} \end{matrix} \right|$$

where $m_r' = m_r + 1$

(This follows because $\left| \begin{matrix} P \\ QP \end{matrix} \right| = \left| \begin{matrix} Q \\ P^2 \end{matrix} \right|$)

$$= L \times \left| \begin{matrix} x^{2^{m_{n+1}}} & x \\ x & 1 \end{matrix} \right| \left| \begin{matrix} x \\ x^{2^{(m_n' \cdots m_1'}} \end{matrix} \right|$$
$$= L \times S_x \frac{(-2)^{2^{m_{n+1}}}-1}{3} \times S_x \frac{1+(-)^{n+1}x2\times(-2)_1{}^{2^{m_1}}+(-2)^{m_2}-+\ldots)}{3}$$
$$= S_x \frac{1+(-)^{n+1}((-2)^{2^{m_1}}+(-2)^{2^{m_2}}-+\ldots)}{3}$$

The theorem is thus true for products of $(n+1)$ elements.

Since the theorem has been proved true for simple powers, the induction shews that it is universally true.

Def. The expression

$$\frac{1+(-)^{m}\left((1-2)^{p_1}+(-2)^{p_2}-(-2)^{p_3}+(-2)^{p_4}\ldots\right)}{3}$$

may be called *the characteristic* of the successive product $x^{q}\,(p_m p_{m-1}\ldots p_1)$. Evidently the characteristic is an intger $+^{ve}$ or $-^{ve}$.

§ 6. A function of x which $=n$ when $x=n'$ will be said to be an even function. A function which $=n'$, when $x=n$ will be saia to be an odd function.

A function which does not become indeterminate when x is made equal to a nodal point will be said to be simplex.

(a) A pure successive product is an even or odd function according as its characteristic is $-^{ve}$ or $+^{ve}$.

To prove this we observe that if $f(x)$ is an odd function $S_x[f(x)]$ is also an odd function; for

$$\begin{aligned} S_x(f(x)) &= x(x^2 f(x)) \\ &= n(n'n') \text{ (putting } x=n) \\ &= n' \end{aligned}$$

Hence $S_x\,(f(x))$ is an odd function. In the same way it is seen that if $f(x)$ is an even function $S_x^{-1}(f(x))$ is also an even function.

Now if P_x be the successive product and k its characteristic.

We have $\left|\begin{matrix} x \\ P_x \end{matrix}\right| = S_x^{k}$ or $P_x = S_x^{k}(x) = S_x^{k-1}(x^2)$

since x is an even function and x^2 an odd function it follows at once that P_x is even or odd according as k is $-^{ve}$ or $+^{ve}$

(b) To find the necessary and sufficient condition that a pure product may be simplex.

Evidently the necessary and sufficient condition is that, in the course of the product an odd function should nowhere be multiplied by an even function or an even function by an odd function.

Let the product be $x^{2}\,(p_m p_{m-1}\ldots p_2 p_1)$

Firstly $x^{2^{p_1}}$ and $x^{2^{p_2}}$ should be both odd or both even functions.

Which shews that p_1 and p_2 should be both odd or both even secondly it should be even or odd according as $a^{2(p_{r-1}p_{r-2}\cdots p_2p_1)}$ is even or odd, for all values of r.

The characteristics are therefore of the same sign.

Therefore $(-)^r(-2)^{p_r}$ is of the same sign as $((-2)^{p_1}+(-2)^{p_2}\ldots \pm(-2)^{p-r1})$

This shews that $p_2p_3p_4\ldots$ are alternately odd and even.

Cor. The degree of a pure simplex product cannot be a multiple of three.

The degree of the product in (b) above is

$$(2)^{p_1}+(2)^{p_2}+2^{(p_3)}\ldots$$

Now $2^{p_1}=\pm1 \pmod 3$

$2^{p_2}=\pm1 \pmod 3$

$2^{p_3}=\mp1 \pmod 3$

$2^{p_4}=\pm1 \pmod 3$

The upper signs being taken if p_1 is even, the lower if p_1 is odd. (These equations follow by applying the conditions that the product may be simplex).

Adding up, we see that the degree can never be a multiple of three.

§ 7. *The General Equation.*

We are now in a position to discuss the general equation

$$F_1(x)=F_2(x) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

F_1, and F_2 can be reduced to successive products [IV] and further reduced to one of the three forms aP_x, $a(bP_x)$, x^{2m} (cP_x) where P_x is a pure product. It is easy to verify that the equation (1) can in all cases be reduced to one of the two types

$$P_x=a \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

$$P_x=a(bQ_x) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

where P_x and Q_x are pure products.

An equation is said to be simplex, when the nodal points are not roots of it. Evidently the equation of the first type is simplex if P_x is simplex and the equation of the second type is simplex if P_x and Q_x are not both odd or both even. We prove from this the interesting result:— *The degree of a simplex equation of the first type is never a multiple of* 3; *the degree of a simplex equation of the second type is always a multiple of* 3.

The first part follows from § 6, Cor.

To prove the second part, we refer back to § 6, Cor.

The sign of the characteristic of the product $x^{\alpha}(P_m P_{m-1} \cdots P_1)$ is $(-)^{p_1+m}$. The degree of the product is of the form $3q+(-)^{m-1}$ if p_1 is even and of the form $(3q+(-)^{m})$ if p_1 is odd; i.e. of the form $(3q-(-)^{p_1+m})$ whether p_1 is odd or even. Hence the degree of a simplex product is of the form $(3q \mp 1)$ according as its characteristic is $+^{ve}$ or $-^{ve}$. Applying this result to the equation of the second type, we see that the degree of the one of the functions P_x, Q_x is of the form $3q+1$, and of the other, of the form $3q-1$, so that the degree of the equation itself is a multiple of 3.

Solution of the General Equation.

(i) *The equation of the first type* $P_x = b$.

Let k be the characteristic of P_x, so that

$$S_x^{k} = \left| \begin{matrix} x \\ P_x \end{matrix} \right| = \left| \begin{matrix} x \\ b \end{matrix} \right| = \left| \begin{matrix} x \\ \omega_1 \end{matrix} \right| \left| \begin{matrix} b \\ \omega_1 \end{matrix} \right| -1$$

Now $S_x = \left| \begin{matrix} x \\ \omega_1 \end{matrix} \right|^3$

$$\therefore \quad \left| \begin{matrix} x \\ \omega_1 \end{matrix} \right|^{3k-1} = \left| \begin{matrix} b \\ \omega_1 \end{matrix} \right| -1 = \text{a known correspondence}$$

This determines x. We see incidentally that the degree of the equation supposed simplex is $3k-1$.

(ii) *The equation of the second type* $P_x = a(bQ_x, = S(Q_x)$

If k, k' are the characteristics of P_x, Q_x, we have

$$S = \left| \begin{matrix} Q_x \\ P_x \end{matrix} \right| = \left| \begin{matrix} x \\ P_x \end{matrix} \right| \left| \begin{matrix} x \\ Q_x \end{matrix} \right| -1$$

$$= S_x^{k-k'} \qquad = \Omega_x^{3(k-k')}$$

which determines Ω_x^{3} and $\therefore x$.

We see that the degree of the equation is the numerical value of $3(k-k')$.

§ 8. The last paragraph shews that the solution of any equation is reduced to the extraction of roots and to the geometrical problem of drawing a ray making with three other rays, a pencil of given cross ratio. It also shews that if α is one root of an equation of the n^{th}

degree, all the roots are given by the series, α, $\Omega_n(\alpha)$, $\Omega^2_n(\alpha)$..... $\Omega_n^{n-1}(\alpha)$, Ω_n being a primitive n^{th} root of the identical correspondence with the nodal points as fixed points.. This is merely another aspect of the fact that in this algebra every equation involves only one parameter. This somewhat strange result is shewn by the theory of linear systems of points to be necessitated by the symmetrical relation between the nodal points and the linear system formed by the roots of all equations of a particular degree. Now there are only two ways in which *two* points can be related in absolute symmetry to a linear system, viz. (1) the system may be a complete one and the two points may form the indeterminate pair, in which case the order of the system must be three; This happens in the case of the nodal points and the complete system of collinear triads; or (2) the system must be a focal system of the second degree involving a single parameter with the two points as foci. The linear system of roots of equations of a particular degree belongs to this latter category.

SHORT NOTES.

Some Fundamental Expansions.*

§ 1. Consider $\frac{\cos\lambda x}{\sin\rho x}$ as the limit, when m becomes infinite, of

$$f(m)=i\frac{\left(1+\frac{i\lambda x}{m}\right)^m+\left(1-\frac{i\lambda x}{m}\right)^m}{\left(1+\frac{i\rho x}{m}\right)^m-\left(1-\frac{i\rho x}{m}\right)^m},$$

and suppose that m reaches infinity through odd positive integral values. Decomposing $f(m)$ into partial fractions, the factors of the denominator are obtained by solving

$$\left(1+\frac{i\rho x}{m}\right)^m-\left(1-\frac{i\rho x}{m}\right)^m=0.$$

This gives

$$\frac{1+\frac{i\rho x}{m}}{1-\frac{i\rho x}{m}}=1^{\frac{1}{m}}=e^{\frac{2ik\pi}{m}} \quad \text{[where } k \text{ is zero or any integer]}$$

$$\therefore \qquad i\rho x=\frac{e^{\frac{ik\pi}{m}}-e^{-\frac{ik\pi}{m}}}{e^{\frac{ik\pi}{m}}+e^{-\frac{ik\pi}{m}}}=i\tan\frac{k\pi}{m}, \text{ i.e. } x=\frac{m}{\rho}\tan\frac{k\pi}{m}.$$

Denoting this value of x by x_k, the partial fractions are

$$\frac{1}{\rho x}+\Sigma\frac{\delta_k}{x-x_k},$$

where by the usual rules

$$\delta_k=\left[\left(1+\frac{i\lambda x_k}{m}\right)^m+\left(1-\frac{i\lambda x_k}{m}\right)^m\right]\div\left[\left(1+\frac{i\rho x_k}{m}\right)^{m-1}+\left(1-\frac{i\rho x_k}{m}\right)^{m-1}\right]$$

and k takes the values, $-\frac{m-1}{2}, -\frac{m-3}{2}, \ldots -1, +1, \ldots \frac{m-1}{2}$,

Also since $x_k=-x_{-k}$ and $\delta_k=\delta_{-k}$,

---

* The four fundamental expansions contained in this Note can be obtained by a method similar to that given in Todhunter's *Integral Calculus*, § 20. The method here given is, perhaps, more rigid.

$$\therefore \qquad f(m)=\frac{1}{\rho x}+\sum \frac{2x\delta_k}{x^2-x_k^2},$$

the summation extending from $k=1$ to $k=\frac{1}{2}(m-1)$.

Now $(1+i\lambda x_k/m)^m+(1-i\lambda x_k/m)^m$

$$=\left(1+i\lambda\tan\frac{k\pi}{m}\Big/\rho\right)^m+\left(1-i\lambda\tan\frac{k\pi}{m}\Big/\rho\right)^m$$

$=(1+i\tan\theta_k)^m+(1-i\tan\theta_k)^m$, putting $\tan\theta_k=\frac{\lambda}{\rho}\tan\frac{k\pi}{m}$,

$=2\sec^m\theta_k\cos m\theta_k$;

and $(1+i\rho x_k/m,)^{m-1}+(1-i\rho x_k/m)^{m-1}$

$$=\left(1+i\tan\frac{k\pi}{m}\right)^{m-1}+\left(1-i\tan\frac{k\pi}{m}\right)^{m-1}$$

$$=2\sec^{m-1}\frac{k\pi}{m}\cos(m-1)\frac{k\pi}{m}$$

$$=2(-)^k\sec^{m-1}\frac{k\pi}{m}\cos\frac{k\pi}{m}.$$

Hence $\delta_k=\frac{(-1)^k}{\rho}\left(\frac{\cos k\pi/m}{\cos\theta_k}\right)\cos m\vartheta_k\Big/\cos^2\frac{k\pi}{m}$.

For any particular large value of m, put

$$f_1(m)=\frac{1}{\rho x}+\sum_{k=1}^{k=p}\frac{2x\delta_k}{x^2-x_k^2},\ \text{and}\ f_2(m)=\sum\frac{2x\delta_k}{x^2-x_k^2};$$

the second summation extending from $k=p+1$ to $k=\frac{1}{2}(m-1)$; then $f(m)=f_1(m)+f_2(m)$. Now let m become infinite; then since $f(m)$ and $f_1(m)$ have each a definite limit, $f_2(m)$ has also a definite limit; also $\underset{m=\infty}{\text{Lt}}\tan\frac{k\pi}{m}=\frac{k\pi}{m}$; and $\underset{m=\infty}{\text{Lt}}\ \delta_k=\frac{(-1)^k}{\rho}\cos\frac{\lambda k\pi}{\rho}$;

and since $\cos\frac{\lambda k\pi}{\rho}\neq 1$, we may suppose p to be taken so great that the numerical value of $f_2(m)$ is less than

$$\sum\frac{2|x|}{|\rho|(k^2\pi^2/\rho^2-x^2)}$$

This last is a convergent series, and therefore when p is taken sufficiently large, $f_2(m)$ can be made less than a quantity as small as we please. Proceeding to the limit when p tends to m or ∞, we get the fundamental form

$$\frac{\cos\lambda x}{\sin\rho x} = \frac{1}{\rho x} + \frac{1}{\rho} \operatorname*{Lt}_{m=\infty} \sum_{k=1}^{k=m} (-1)^k \cos\frac{\lambda k\pi}{\rho} \frac{}{(x^2 - k^2\pi^2/\rho^2)} 2x. \quad \ldots \quad \ldots \quad (A)$$

In particular if $\rho = 1 = \lambda$, we have

$$\cot x = \frac{1}{x} + \operatorname*{Lt}_{m=\infty} \sum_{k=1}^{k=m} \frac{(-1)^k \cos k\pi}{(x^2 - k^2\pi^2)} 2x = \frac{1}{x} + \sum_{k=1}^{k=\infty} \frac{2x}{x^2 - k^2\pi^2},$$

as is well known ; and by putting $\lambda = 0$, $\rho = 1$, we get

$$\operatorname{cosec} x = \frac{1}{x} + \sum_{k=1}^{k=\infty} \frac{(-1)^k}{x^2 - k^2\pi^2} 2x.$$

We can similarly find out the expansions of the remaining three forms, namely

$$\frac{\sin\lambda x}{\sin\rho x}, \frac{\cos\lambda x}{\cos\rho x}, \frac{\sin\lambda x}{\cos\rho x}$$

by reasoning as for the first fundamental form : the following convergent expansions will be obtained :—

$$\frac{\sin\lambda x}{\sin\rho x} = \sum_{k=1}^{k=\infty} \frac{(-1)^k 2\pi k \sin\frac{\lambda k\pi}{\rho}}{(\rho^2 x^2 - k^2\pi^2)}, \quad [\lambda < \rho],$$

$$= \operatorname*{Lt}_{m=\infty} \sum_{k=-m}^{k=+m} \frac{(-1)^k \sin\frac{\lambda k\pi}{\rho}}{(\rho x - k\pi)}; \quad \ldots \quad \ldots \quad (B)$$

$$\frac{\cos\lambda x}{\cos\rho x} = \sum_{k=1}^{k=8} \frac{(-1)\cos\frac{\lambda}{\rho}(2k-1)\frac{\pi}{2}}{x^2 - (2k-1)\frac{\pi^2}{4\rho^2}} \cdot \frac{(2k-1)\pi}{\rho^2}, \quad [\lambda < \rho]$$

$$= \sum_{k=1}^{k=\infty} \frac{(-1)^k \cos\frac{\lambda}{\rho}(2k-1)\frac{\pi}{2}}{\rho^2 x^2 - (2k-1)^2\frac{\pi^2}{4}} \cdot (2k-1)\pi; \quad \ldots \quad (C)$$

and

$$\frac{\sin\lambda x}{\cos\rho x} = \sum_{k=1}^{k=\infty} \frac{(-1)^k \rho \sin\frac{\lambda}{\rho}(2k-1)\frac{\pi}{2}}{\rho^2 x^2 - (2k-1)^2\frac{\pi^2}{4}} 2x, \quad [\lambda < \rho]$$

$$= \operatorname*{Lt}_{m=\infty} \sum_{k=-m}^{k=+m} \frac{(-1)^k \sin(2k-1)\frac{\lambda\pi}{2\rho}}{\rho x - (2k-1)\frac{\pi}{2}} \rho. \quad \ldots \quad \ldots \quad (D)$$

In (B) and (D), the value $k=0$ should not be included.

These results may also be obtained from (A) by observing

$$\frac{\sin\lambda x}{\mathrm{sn}x}=\frac{1}{\sin\lambda\pi}\left\{\cos\lambda\pi\,\frac{\cos\lambda x}{\sin x}+\frac{\cos\lambda(x+\pi)}{\sin(x+\pi)}\right\}$$

and using similar trigonometrical identities.

Taking the first fundamental form and changing λ to $i\lambda$, we get

$$\frac{\cosh\lambda x}{\sin\rho x}=\frac{1}{\rho x}+\frac{1}{\rho}\sum_{k=1}^{k=\infty}{}'\frac{(-1)^k\cosh\dfrac{k\lambda\pi}{\rho}}{(x^2-k^2\pi^2/\rho^2)}\,2x. \quad \ldots \quad \ldots \quad (\mathrm{A}')$$

Further changing ρ to $i\rho$, we get

$$\frac{\cos\lambda x}{\sinh\rho x}=\frac{1}{\rho x}+\frac{1}{\rho}\sum_{k=1}^{k=\infty}{}'\frac{(-1)^k\cosh\dfrac{\lambda k\pi}{\rho}}{(x^2+k^2\pi^2/\rho^2)}\,2x. \quad \ldots \quad \ldots \quad (\mathrm{A}'')$$

Lastly changing λ to $i\lambda$ and ρ to $i\rho$, we get

$$\frac{\cosh\lambda x}{\sinh\rho x}=\frac{1}{\rho x}+\frac{1}{\rho}\sum_{k=1}^{k=\infty}{}'\frac{(-1)^k\cos\dfrac{k\lambda\pi}{\rho}}{(x^2+k^2\pi^2/\rho^2)}\,2x. \quad \ldots \quad \ldots \quad (\mathrm{A}''')$$

Taking the second fundamental form (B) and making similar changes, we shall have—

$$\frac{\sinh\lambda x}{\sin\rho x}=\sum_{k=1}^{k=\infty}{}'\frac{(-1)^k 2\pi k\sinh\dfrac{\lambda k\pi}{\rho}}{\rho^2x^2-k^2\pi^2}, \quad \ldots \quad \ldots \quad (\mathrm{B}')$$

$$\frac{\sin\lambda x}{\sinh\rho x}=\sum_{k=1}^{k=\infty}{}'\frac{(-1)^{k-1}2\pi k\sinh\dfrac{\lambda k\pi}{\rho}}{\rho^2x^2+k^2\pi^2}, \quad \ldots \quad (\mathrm{B}'')$$

and

$$\frac{\sinh\lambda x}{\sinh\rho x}=\sum_{k=1}^{k=\infty}{}'\frac{(-1)^{k-1}2\pi k\sin\dfrac{\lambda k\pi}{\rho}}{\rho^2x^2+k^2\pi^2}. \quad \ldots \quad \ldots \quad (\mathrm{B}''')$$

Similarly from the third fundamental form (C) we get

$$\frac{\cosh\lambda x}{\cos\rho x}=\sum_{k=1}^{k=\infty}{}'\frac{(-1)^k\cosh\dfrac{\lambda}{\rho}(2k-1)\dfrac{\pi}{2}}{x^2-(2k-1)^2\dfrac{\pi}{4\rho^2}}(2k-1)\frac{\pi}{\rho^2}, \quad \ldots \quad (\mathrm{C}')$$

and from the fourth fundamental form (D)

$$\frac{\sinh\lambda x}{\cos\rho x}=\sum_{k=1}^{k=\infty}\frac{(-1)^k\rho\sinh\frac{\lambda}{\rho}(2k-1)\frac{\pi}{2}}{\rho^2x^2-(2k-1)^2\frac{\pi^2}{4}}\,2x. \qquad \ldots \qquad \ldots \qquad (D')$$

§ 2. A large number of results may be obtained from these forms by various analytical methods.

For example :—

If $z=\sin\rho x, \frac{\sin\lambda x}{\sin\rho x}=\frac{1}{z}\sin\left(\frac{\lambda}{\rho}\sin^{-1}z\right)$; hence by the ordinary expansion we have

$$\frac{\sin\lambda x}{\sin\rho x}=\frac{\lambda}{\rho}\left\{1+\frac{1}{3!}\left(1-\frac{\lambda^2}{\rho^2}\right)\sin^2\rho x + \frac{1}{5!}\left(1-\frac{\lambda^2}{\rho^2}\right)\left(3^2-\frac{\lambda^2}{\rho^2}\right)\sin^4\rho x+\ldots\right\}$$

$$=\frac{2}{\pi}\sin\frac{\lambda\pi}{\rho}\left(1-\rho^2x^2/\pi^2\right)^{-1}-\frac{4}{2^2\pi}\sin\frac{2\lambda\pi}{\rho}$$

$$(1-\rho^2x^2/2^2\pi^2)^{-1}+\ldots$$ by resnlt (B); from which, by equating coefficents of different powers of x, we derive the following series

$$\sin\frac{\lambda\pi}{\rho}-\frac{1}{2}\sin\frac{2\lambda\pi}{\rho}+\frac{1}{3}\sin\frac{3\lambda\pi}{\rho}-\ldots=\frac{\pi}{2}\cdot\frac{\lambda}{\rho} \qquad \ldots \qquad \ldots \qquad (1)$$

$$\frac{1}{1^3}\sin\frac{\lambda\pi}{\rho}-\frac{1}{2^3}\sin\frac{2\lambda\pi}{\rho}+\frac{1}{3^3}\sin\frac{3\lambda\pi}{\rho}-\ldots=\frac{\pi^3}{1^2}\left(\frac{\lambda}{\rho}-\frac{\lambda^3}{\rho^3}\right) \qquad \ldots \qquad (2)$$

$$\frac{1}{1^5}\sin\frac{\lambda\pi}{\rho}-\frac{1}{2^5}\sin\frac{2\lambda\pi}{\rho}+\frac{1}{3^5}\sin\frac{3\lambda\pi}{\rho} =\frac{\pi^5}{720}\frac{\lambda}{\rho}\left(1-\frac{\lambda^2}{\rho^2}\right)\left(1-\frac{3\lambda^2}{\rho^2}\right) \qquad \ldots \qquad (3)$$

$$\frac{1}{1^7}\sin\frac{\lambda\pi}{\rho}-\frac{1}{2^7}\sin\frac{2\lambda\pi}{\rho}+\frac{1}{3^7}\sin\frac{3\lambda\pi}{\rho}- =\frac{\pi^7}{2}\frac{\lambda}{\rho}\left(1-\frac{\lambda^2}{\rho^2}\right)\left\{\frac{2}{45}-\frac{1}{240}\left(3^2-\frac{\lambda^2}{\rho^2}\right) + \frac{1}{7!}\left(3^2-\frac{\lambda^2}{\rho^2}\right)\left(5^2-\frac{\lambda^2}{\rho^2}\right)\right\} \qquad \ldots \qquad \ldots \qquad (4)$$

The series (1), (2), (3), (4), are all convergent.

[See : Chrystal. Chap XXVI, §9]

M. K. KEWALRAMANI.

An Interesting Experiment.

[The following interesting experiment described in December 29, 1917 issue of "Tit-Bits" was successfully performed by the present writer, and is reported here in view of its scientific interest. The details of the experiment can be varied, at pleasure, of course.]

Take a good-sized bowl, fill it nearly with water, and place it upon the floor of a room which is not exposed to shaking or jarring from the street traffic. Sprinkle over the surface of the water a coating of lycopodium powder, which can be obtained at any hospital or chemists' shop. Then upon the surface of this coating of powder make with powdered charcoal a straight black line, say an inch or two in length. Having made this little mark with the charcoal powder on the surface of the contents of the bowl, lay down upon the floor close to the bowl a stick (or some other straight object) so that it will be exactly parallel to the mark. (If the line happens to be parallel to a crack in the floor, or to any stationary object in the room, this will serve as well.)

Leave the bowl undisturbed for a few hours, and then observe the position of the black mark, with reference to the object to which it was parallel. It will be found to have moved in the direction opposite to that of the movement of the earth on its axis The earth is simply rotating and has carried the water and everything else in the bowl round with it, but the powder on the surface has been left behind a little. The line will always be found to have moved from east to west, which is a perfectly good proof that everything else has moved the other way.

K. B. M.

An interesting Cubic Connected with an Ellipse.

Basset in his *Treatise on Cubic and Quartic Curves* has discussed the different forms of the cubics obtained by inverting a conic with respect to its vertex. (vide : §§ 125, 146-7). The following interesting case has however been overlooked by him :

Let an ellipse be referred to the extremity of its minor axis as origin. Then its equation is written

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{2y}{b}$$

Inverting with respect to the origin, we have the circular cubic

$$2y\frac{(x^2+y^2)}{b}=k^2\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right),$$

whose form is shown below.

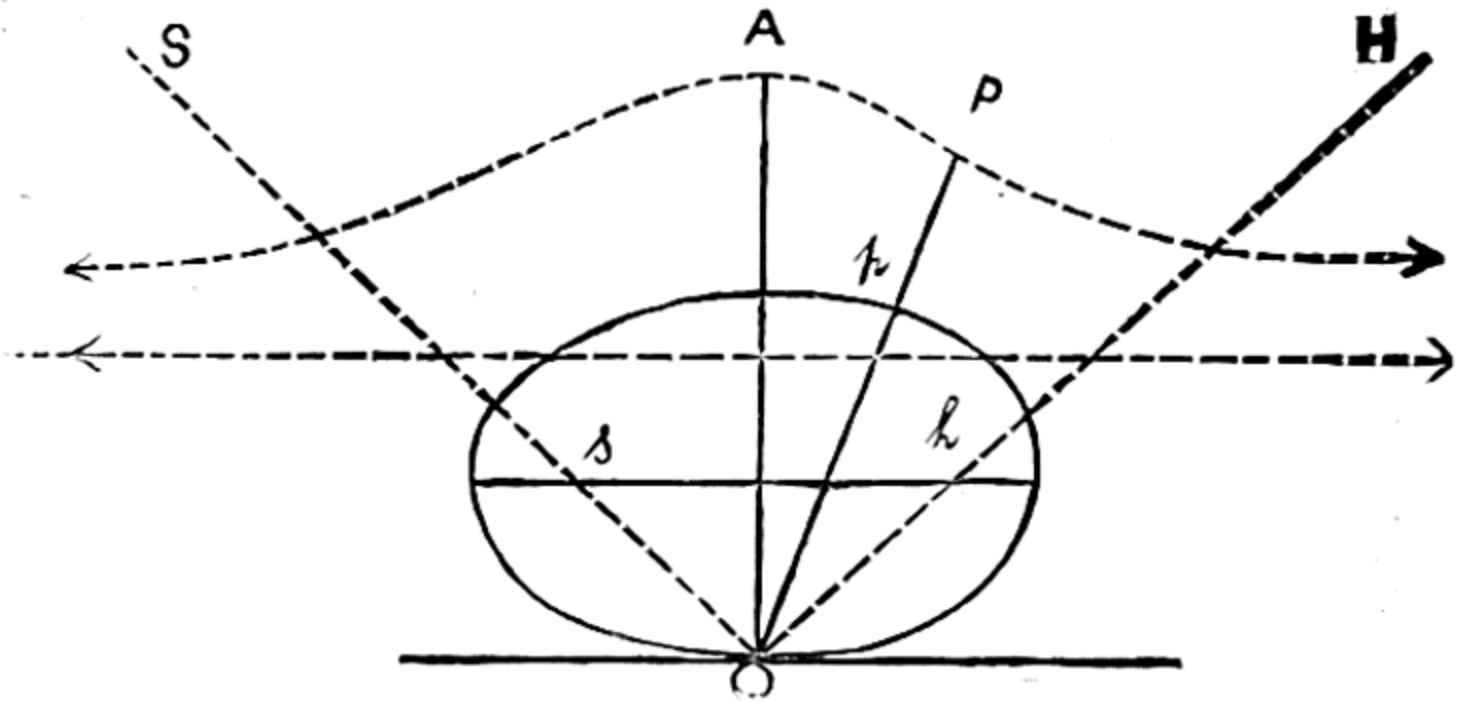


Fig. 1.

If S, H be the inverses of the foci s, h of the ellipse. and P, p be any two corresponding points on the cubic and the ellipse, we readily obtain,

$$\frac{SP}{OS}+\frac{HP}{OH}=\frac{OP}{k^2}(sp+hp)$$

$$=2a.\frac{OP}{k^2}.$$

But $os=oh=a,\ OS\cdot os=k^2=OH.oh.$

$\therefore$ $OH=OS=k^2/a.$

Hence $SP+HP=2\cdot OP,$

In other words, the tri-polar equation of the cubic is

$$r_1+r_2=2r_3.$$

M. T. Naraniengar.

Degenerate Bicircular Quartics.

The equation of two inverse circles (the origin being the centre of inversion) is

$$(r^2+2cx+\delta^2\lambda)(r^2+2cx/\lambda+\delta^2/\lambda)=0.$$

Comparing this with equation (12) on p. 136 of Basset: *Cubic and Quartic Curves*, we infer that a bicircular quartic will degenerate in two inverse circles, if

$$4f = 2c\left(1+\frac{1}{\lambda}\right),\ g=0,$$

and
$$4f^2+2\delta^2-4a^2=\frac{4c^2}{\lambda}+\delta^2\left(\lambda+\frac{1}{\lambda}\right),$$

$$2\delta^2-4b^2=\delta^2\left(\lambda+\frac{1}{\lambda}\right).$$

Thus
$$f^2=c^2/\lambda+(a^2-b^2)$$

$$=\frac{4f^2\lambda}{(1+\lambda)^2}+(a^2-b^2)\ ;$$

also
$$\delta^2\left(\lambda+\frac{1}{\lambda}+2\right)=4(\delta^2-b^2).$$

$$\therefore\quad f^2\left(1-\frac{\delta^2}{\delta^2-b^2}\right)=a^2-b^2.$$

$$\therefore\quad f^2=\frac{(a^2-b^2)(b^2-\delta^2)}{b^2}$$

$$\therefore\quad f=\pm\sqrt{\left\{\frac{(a^2-b^2)(b^2-\delta^2)}{b^2}\right\}}. \qquad \ldots \qquad \ldots \quad (1)$$

In other words, if the centre of inversion is the point $(f, 0)$ where f is determined by (1), the bicircular quartic reduces to two inverse circles.

Examples :

(i) Let the centre of inversion be the foot of the normal at P(ϕ) ; then

$$f=\frac{(a^2-b^2)}{a}\cos\phi\ ;$$

$$\therefore\quad \left(1-\frac{b^2}{a^2}\right)\cos\phi=\left(1-\frac{\delta^2}{b}\right)\text{ by (1).}$$

$$\therefore\quad \delta^2=\frac{b^2}{a^2}(a^2\sin^2\phi+b^2\cos^2\phi)=\frac{b^2.\mathrm{CD}^2}{a^2},$$

$$=\mathrm{PG}^2.$$

Hence, we have the following result :

If from G a perpendicular GT be drawn on any tangent and a point R taken on this perpendicular such that $\mathrm{GT}^2-\mathrm{RT}^2=\mathrm{PG}^2$, then the locus of R is either of two circles.

(ii) Suppose $\phi=0$ in the above. Then $f=(a^2-b^2)/a$, and $\delta=b^2/a$.

For points outside the limits $\pm\ (a^2-b^2)/a$, the value of δ is real up to $\sqrt{(a^2-b^2)}$, and beyond that δ becomes imaginary, being the length of the imaginary normal from the point to the ellipse.

M. T. Naraniengar.

Astronomical Notes.

The New Star in Aquila.

The new star in Aquila appeared very suddenly on June 8 last when it was about the same magnitude as Altair, for about two days it increased in magnitude till it became nearly equal to Vega and then began to fade steadily. At first the star was white in colour, but about five days after its first appearance it became distinctly yellow, and a few days later became red. The spectrum about this stage showed a velocity of approach of about one-thousand miles per second, similar to that observed in Nova Persei.

The Nova appeared in the dark rift in the Milky way about one hour west of Altair, its approximate position being R.A. $18^h\ 44^m\ 47^s$ Decl. 0° 31′ N.

This is the fourth Nova which has been discovered in Aquila. In 389 A.D. a bright Nova appeared close to Altair, but vanished after three weeks In 1899 and again in 1905 photographic discoveries of Nova in Aquila were made at Harvard College Observatory. The position of the present Nova however differs greatly from any of the other three so that identity is not possible.

An interesting Problem.

The sixth and seventh satellites of Jupiter revolve about their primary at approximately the same distance and consequently in approximately the same period. The same is true of the eighth and ninth satellites, the sidereal periods of which are 739 days and 745 days respectively, a difference of less than one per cent. Now it appears that the perijories of the sixth and seventh satellites are separated by approximately 180°, while the same is found to be true of the eighth and ninth satellites and the question arises, is this purely accidental or is there any dynamical reason? for example, is this arrangement more stable than any other?

It may be mentioned that whereas the motion of the sixth and seventh satellites is direct that of the eighth and ninth is retrograde and it was shown a few years ago that at a considerable distance from the primary a retrograde satellite is more stable than a direct satellite having the same mean distance.

Relativity.

A great deal continues to be published on the subject of Relativity, both with respect to Einstein's recent theory and Sir Oliver Lodge's attempt to explain the motion of the perihelion of Mercury, theories, both of which have been previously referred to in this Journal. I cannot give all the references but the following paper may be found of interest by those interested in the subject.

Planetary motion in space—Time of any constant curvature according to the generalised principle of relativity; by L. Silberstein. *Mon. Not. R. A. S.* 1918 March, p. 363.

R. J. Pocock.

SOLUTIONS.

Question 612.

(N. Ganapathi Subba Aiyar):—A series of ellipses have a common focus (S), pass through a given point (P) and have their major axes of the same length. Find their envelope.

Additional Solution by N. Durairajan.

Lemma: If two curves touch, their first positive pedals touch. Therefore if a number of curves have a common envelope the pedals of the curves have also a common envelope which is the the pedal of the original envelope.

In the present case, the pedals of the ellipses are circles of constant radii and also the circles on SP as diameter touches these circles. The envelope of these circles clearly consists of two circles of radii $2a+2r$, where a denotes the semi-major axis and r denotes SP.

Hence the envelope of the ellipses is the negative pedal of these two circles and consists of two conics having a common focus S.

Question 625.

(V. V. S. Narayan):—Show that the lines joining the vertices of a triangle to the corresponding vertices of its first Brocard triangle concur in a point Q whose trilinear co-ordinates are inversely proportional to those of the pole of $\Omega\,\Omega'$ with respect to the Brocard circle,

Solution by K. B. Madhava.

Let K be the symmedian point, and EF', FD', DE' be the antiparallels though it cutting the circle on SK (S is the circumcentre) as diameter at the points A', B', C' respectively. Then A'B'C' is said to be the first Brocard triangle and the circle as the Brocard circle.

Its equation in trilinears will be seen to be (by the method of Askwith *Coordinate Geometry*: p. 416. Ex. 4,).

$$\alpha^2+\beta^2+\gamma^2-\frac{a^2}{bc}\beta\gamma-\frac{b^2}{ca}\gamma\alpha-\frac{c^2}{ab}\alpha\beta=0.$$

Since the first Lemoine circle goes through EF' and is concentric with the Brocard circle through K, A'; KE = A'F'; *i.e.* AA' and AK

ent BC in points equidistant from its midpoint. Therefore by Ceva's theorem AK′, BB′, CC′ concur and their point of intersection (Q) is the isotomic conjugate of K, *i.e.* its trilinear coordinates are

$$\left(\frac{1}{a^3}, \frac{1}{b^3}, \frac{1}{c^3}\right).$$

Moreover since the normal coordinates of

$$\Omega \text{ are } \frac{c}{b}, \frac{a}{c}, \frac{b}{a} \text{ and of } \Omega^1 \quad \frac{b}{c}, \frac{c}{a}, \frac{a}{b} \qquad \ldots \qquad \ldots \qquad (2)$$

the equation of $\Omega\ \Omega^1$ is

$$\alpha\left(a^3-\frac{b^2c^2}{a}\right)+\beta\left(b^3-\frac{a^2c^2}{b}\right)+\gamma\left(c^3-\frac{b^2a^2}{c}\right)=0 \qquad \ldots \qquad (3)$$

and its pole with respect to the circle (1) is (a^3,b^3,c^3). Hence the result stated.

Question 734.

(J. C. SWAMINARAYAN) :—Solve the differential equation

$$(y^2-b^2)\frac{dy}{dx}+xy=\sqrt{b^2x^2+a^2y^2-a^2b^2}.$$

Solution by N. Durairajan.

Write $x=a\cos(\alpha\cdot v.)$

$y=b\sin(\alpha\cdot v.)$

Substituting in the equation and reducing we get

$$d\alpha+dv.\frac{\sqrt{v^2-1}}{v}=0.$$

The variables are separated and the equation may be considered as solved.

Question 753.

(S. RAMANUJAM) :—If

$$\phi(x)=\tfrac{1}{2}\log 2\pi x-x+\int_1^x\frac{[t]}{t}dt$$

where $[t]$ denotes the greatest integer in t, show that

$$\overline{\lim_{x\to\infty}}\ x\phi(x)=\frac{1}{24} \text{ and } \frac{\lim}{x\to\infty}\ x\phi(x)=-\frac{1}{12}.$$

Solution (1) *by N. Durairajan* ; (2) *by K. B. Madhava.*

(1) We have

$$\phi(x)=\frac{1}{2}\log 2\pi x-x+\int_1^x\frac{[t]}{t}dt.$$

Expressing the integral as I, we have

$$I=\int_1^2 \frac{[t]}{t}\,dt+\int_2^3 \frac{[t]}{t}\,dt+\ldots+\int_{[x]}^{x} \frac{[t]}{t}dt.$$

$$=1\log\frac{2}{1}+2\log\frac{3}{2}+3\log\frac{4}{3}+\ldots+[x]\log\frac{x}{[x]}.$$

$$=\log\left[\frac{2}{1}\cdot\frac{3^2}{2^2}\cdot\frac{4^3}{3^3}\cdots\frac{x^{(x)}}{[x]^{(x)}}\right]$$

$$=\log\frac{x^{(x)}}{[x]\,!}$$

We have

$$\phi(x)=\log\frac{e^{-x}\sqrt{2\pi x}.\ x^{(x)}}{[x]\,!}$$

Now let $x=[x]+\delta=n+\delta$, and keeping δ constant make $n\to\infty$.

Now

$$\lim_{n\to\infty}(x\phi(x))=\lim_{n\to\infty}(n+\delta)\log\frac{e^{-n-\delta}\sqrt{2\pi(n+\delta)}(n+\delta)^n}{n\,!}$$

δ being constant.

$$=\lim_{n\to\infty}(n+\delta)\log\frac{e^{-n}\sqrt{2\pi n}.n^n\left(1+\frac{\delta}{n}\right)^{n+\frac{1}{2}}e^{-\delta}}{n\,!}$$

$$=\lim_{n\to\infty}(n+\delta)\log e^{-\frac{1}{12n}-\theta}$$

$$+(n+\delta)\log\left\{e^{-\delta}\left(1+\frac{\delta}{n}\right)^{n+\frac{1}{2}}\right\}$$

by Stirling's theorem, where $\theta=0\left(\frac{1}{n^2}\right)$:

$$=(n+\delta)\left(-\frac{1}{12}-\theta\right)+(n+\delta)\left(\frac{\delta-\delta^2}{2n}+\frac{4\delta^3-3\delta^2}{12n^2}+\ldots\right)$$

$$=-\frac{1}{12}+\frac{1}{2}(\delta-\delta^2)-\frac{\delta}{12n}(1-6\delta+2\delta^2)+\ldots$$

In other words, the limit depends on δ. The maximum and minimum values of $\delta-\delta^2$ are when $\delta=\frac{1}{2}$ and 0, and equal to $\frac{1}{4}$ and 0 respectively. Hence $\lim x\phi(x)$ oscillates between the limits

$$-\frac{1}{12}+\frac{1}{2}\frac{1}{4} \quad \text{and} \quad -\frac{1}{12}+0 \ \textit{ie} \text{ between } \frac{1}{24} \text{ and } -\frac{1}{12},$$

(2) Differentiating we have

$$\frac{d\phi}{dx}=\frac{1}{2x}-1+\frac{n}{x},$$

which vanishes when $x=n+\frac{1}{2}$.

$$\text{Also } \frac{d^2\phi}{dx^2}=-\frac{1+2n}{2x^2}=\text{negative when } n\to\infty$$

Hence we infer that

$$\overline{\lim_{x\to\infty}}\, x\phi(x)=\lim_{n\to\infty}\ (n+\tfrac{1}{2})\phi(n+\tfrac{1}{2}).$$

(Vide Solution to Q 740, page 220., Vol. VIII)

Question 783.

(S. RAMANUJAN) :—If $x=y^n-y^{n-1}$, and

$$J_n=\int_0^1 \log y\,\frac{dx}{x},$$

show that

(i) $J_0=\frac{\pi^2}{6}$; $J_{\frac{1}{2}}=\frac{\pi^2}{10}$; $J_1=\frac{\pi^2}{12}$; $J_2=\frac{\pi^2}{15}$.

(ii) $J_n+J_{\frac{1}{n}}=\frac{\pi^2}{6}$.

Solution by N. Durai Rajan and 'Zero'

(I) If $x=y^n-y^{n-1}$, it is obvious that for positive values of x, $y>1$ when positive. Also, y cannot be negative, since the question involves $\log y$. Thus, the values of y corresponding to the limits (0 , 1) of x are (1 , α), where

$$\alpha^n-\alpha^{n-1}-1=0 \quad \text{and } \alpha>1.$$

Again, if $x=y^{\frac{1}{n}}-y^{\frac{1}{n}-1}$, the similar limits for y are $(1,\beta)$, where

$$\beta^{\frac{1}{n}}-\beta^{\frac{1}{n}-1}-1=0$$

which may be written (after multiplication by $\beta^{1-\frac{1}{n}}$)

$$\beta-\beta^{1-\frac{1}{n}}-1=0$$

proving that $\beta=\alpha^n$, and $\frac{1}{\alpha}+\frac{1}{\beta}=1$. ... ... (1)

Now

$$J_n=\int_0^1 \log y\, d(\log x),$$

$$=[\log x \log y]-\int_1^\alpha \log x\, d(\log y),$$

$$J_n=-\int_1^\alpha \left\{ n \log y+\log\left(1-\frac{1}{y}\right)\right\}\frac{dy}{y},$$

since the value of $(\log x \log y)$ between the limiting values of x and y is easily seen to be zero.

Hence

$$J_n=-\left\{\frac{n}{2}(\log y)^2+\int\log\left(1-\frac{1}{y}\right)\frac{dy}{y}\right\}_1^\alpha$$

$$=-\tfrac{1}{2}(\log\alpha \log\beta)-\int_1^\alpha \log\left(1-\frac{1}{y}\right)\frac{dy}{y}. \quad \ldots \quad (2)$$

Similarly

$$J_{\frac{1}{n}}=-\frac{1}{2}(\log\alpha\log\beta)-\int_1^\beta \log\left(1-\frac{1}{y}\right)\frac{dy}{y}. \quad \ldots \quad \ldots \quad (3)$$

To prove that $J_n+J_{\frac{1}{n}}$ is constant, we differentiate (2) and (3) with respect to α and β and remember that

$$\frac{d\alpha}{\alpha^2}+\frac{d\beta}{\beta^2}=0 \text{ by virtue of (1.)}$$

$$\frac{1}{\alpha}+\frac{1}{\beta}=1.$$

Thus

$$\frac{dJ_n}{d\alpha}=-\frac{1}{2}\left\{\frac{\log\beta}{\alpha}+\frac{\log\alpha}{\beta}\frac{d\beta}{d\alpha}\right\}-\frac{1}{\alpha}\log\left(1-\frac{1}{\alpha}\right)$$

$$=-\frac{1}{2}\frac{\log\beta}{\alpha}+\frac{\beta\log\alpha}{2\alpha^2}+\frac{\log\beta}{\alpha}$$

$$=\frac{(\alpha\log\beta+\beta\log\alpha)}{2\alpha^2};$$

and

$$\frac{dJ_{\frac{1}{n}}}{d\beta}=\frac{(\alpha\log\beta+\beta\log\alpha)}{2\beta^2}.$$

$\therefore$

$$\frac{d\left(J_n+J_{\frac{1}{n}}\right)}{d\alpha}=0, \text{ since } \frac{d\alpha}{\alpha^2}+\frac{d\beta}{\beta^2}=0.$$

Hence $J_n+J\frac{1}{n}$ is a constant for all values of n.

In particular $J_1+J_1=2\left(\frac{\pi^2}{12}\right)=\frac{\pi^2}{6}$, by the first part. Finally, therefore, we have

$$J_n+J_{\frac{1}{n}}=\frac{\pi}{6},$$

II. Since $x=y^n-y^{n-1}=y^n\left(1-\frac{1}{y}\right)$ is transformed, by the substitution $\frac{1}{\eta}=1-\frac{1}{y}$, into

$$x^{\frac{1}{n}}=\left(1-\frac{1}{\eta}\right)\ \eta^{\frac{1}{n}},$$

we see that

$$J_{\frac{1}{n}}=\int_0^1\frac{d\xi}{\xi}\log\eta, \text{ where } \xi=x-\frac{1}{n},\ \frac{1}{\eta}=1-\frac{1}{y}.$$

Thus $J_{\frac{1}{n}}=\int_\infty^1\frac{1}{n}\frac{dx}{x}\log\left(1-\frac{1}{y}\right)$.

$$\therefore J_n+J_{\frac{1}{n}}=\int_0^1\frac{dx}{x}\log y+\int_\infty^1\frac{1}{n}\frac{dx}{x}\log\left(1-\frac{1}{y}\right)$$

$$=\int_0^\infty\frac{dx}{x}\log y+\int_\infty^1\left\{\frac{dx}{x}\log y+\frac{dx}{nx}\log\left(1-\frac{1}{y}\right)\right\}$$

$$=\int_1^\infty\left[n\frac{dy}{y}+d\log\left(1-\frac{1}{y}\right)\right]\log y+\int_\infty^1\frac{\log x}{n}d(\log x)$$

$$=\left[\frac{n}{2}(\log y)^2-\frac{1}{2n}(\log x)^2\right]_1^\infty+\int_1^\infty\log y\,d\log\left(1-\frac{1}{y}\right)$$

$$=\left[\log y\left(1-\frac{1}{y}\right)-\frac{1}{2n}\left(\log 1-\frac{1}{y}\right)^2-\log y\left(1-\frac{1}{y}\right)\right]_1^\infty$$

$$-\int_1^\infty\log\left(1-\frac{1}{y}\right)\frac{dy}{y}$$

$$=\frac{\pi^2}{6},$$

since the expression within the square brackets is zero.

Question 800.

(S. Malhari Rao) :—Shew that the sum of all fractions which may be represented by a recurring decimal of the form $(.\dot{a}\ b\ c\ d)$ is 50, provided $a+c=b+d=9$.

Further Remarks by H. Br.

The error in printing the question on p. 199 of Vol. IX was fortunate; and so was the failure of the attempt on p. 251 of the same volume to correct this error. For, the solutions on p. 326 of the last issue of the Journal show that the second dot might be placed on a or on b without altering the proposition. On looking more closely into the matter I find that the second dot might also have been placed on c or nowhere at all. The moral credit for the following generalization thus belongs to the Printer's Devil.

If $(.ab\ldots g\dot{h}\ldots\dot{n})$ is a recurring fraction (pure or mixed) its value is equal to

$$Aa+Bb+\ldots+Nn$$

where the quantities A, B,...N are determined by the positions of the corresponding digits, but are independent of the value of those digits. As a general rule A, B,...N vary with the variations in the form of the recurring fraction, but for all such fractions

$$A+B+\ldots+N=\frac{1}{9}$$

as is clear from the fact that the expression on the left is the value of the continued fraction when all the digits are equal to unity.

The number of the digits and the position of the dots being fixed, let us obtain all possible recurring fractions as under :—

(i) The digits in k assigned positions are given any value we please, independently of each other;

(ii) Each digit in any of the other positions is dependent on a digit in the first group in such a way that there is a complete 1–1 correspondence between the ten possible values of the two. Subject to this limitation, the law of dependence is arbitrary and not necessarily the same for any two pairs, and each of the independent digits may have any number (from zero upwards) of digits depending upon it.

The proposition is that the sum of all fractions is $\frac{1}{2}.10^{-k}$

Proof :—The sum is

$$A\Sigma a + B\Sigma b + \ldots + N\Sigma n$$

where the sammation extends over all admissible values of each digit.

But the 1–1 correspondence ensures that the summation for each of the two connected digits is the same. Clearly the summation for all the independent digits is the same, *i.e.*

$$10^{k-1}.(0+1+2+\ldots+9) = 10^{k-1} \times 45 = \frac{9}{2}.\ 10^{k}.$$

Therefore the sum of all fractions is

$$\frac{9}{2}.10^{k}.(A+B+\ldots+N) = \frac{1}{2}10^{k}.$$

Whatever k be, the average value of the fractions is $\frac{1}{2}$ as it should be because the average value of *all* possible fractions is $\frac{1}{2}$, and we have taken a fair sample of *all* fractions.

Question 925.

(S. R. Ranganathan) :—If P, N, N_0, N_1, are the corresponding points on a curve, its evolute, the evolute of the evolute and the evolute of the α—evolute respectively, show that N_1 describes a circle on N_0 N as diameter, as α varies from $-\frac{\pi}{2}$ to $+\frac{\pi}{2}$.

Solution by R. Srinivasan, M. A., M. R. A. S.

Let P, P^1, P^{11} be consecutive points on the curve, N, N^1 on the evolute ;

then $N\hat{P}N_1 = N\hat{P}^1N_1 = \alpha$.

$\therefore$ $N\hat{N}_1P_1 = N\hat{P}P^1 = 90°$ ultimately.

$\therefore$ $N_1\hat{N}N_0 = 90° - N_1\hat{N}P^1 = \alpha$.

Similarly $N_1\hat{N}^1N_0 = \alpha$ ultimately.

$\therefore$ N_1 N N^1 N_0 is cyclic.

$\therefore$ $N\hat{N}_1N_0 = 180° - N\hat{N}^1N_0 = 90°$ ultimately

$\therefore$ N_1 lies on the circle on N N_0 as diameter.

As α (*ie* $N_1\hat{N}N_0$) varies from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$, N_1 will describe the circle completely.

Question 866.

(H. R. KAPADIA) :—Prove that

$$\int_{\infty} x^{-2n} \sum_{1}^{n}(A_r \cos a_r x)\, dx = \frac{\pi \cos n\pi}{(2n-1)} \sum_{1}^{n} (A_r a_r^{2n-1}).$$

Solution by K. B. Madhava, V. D. Gokhale, Sadanand, M, K. Kewalramani, S. V. Venkatachela Iyer, and K. R. Rama Iyer.

Since the integrand is even, it is equal to twice the value of

$$\int_0^{\infty} x^{-2n} \sum_{1}^{n} (A_r \cos a_r x)\, dx.$$

Integrating this by parts we are successively led to

$$\int_0^{\infty} \frac{1}{2n-1} \sum_{1}^{n} A_r a_r (\sin a_r x) \frac{dx}{x^{2n-1}}$$

and $$\int_0^{\infty} \frac{-1}{(2n-1)(2n-2)(2n-3)} \sum_{1}^{n} A_r a_r^3 \sin a_r x) \frac{dx}{x^{2n-3}}$$

and so on;

assuming $\Sigma A_r = 0$; $\Sigma A_r a_r^2 = 0$ up to

$$\int_0^{\infty} \frac{(-)^n}{(2n-1)!} \Sigma(A_r a_r^{2n-1}) \frac{\sin x}{x} dx$$

$$= \frac{\pi}{2} \cdot \frac{\cos n\pi}{(2n-1)!} \Sigma(A_r a_r^{2n-1}).$$

Hence the result.

Note: It is easy to see that the allied integral

$$\int_0^{\infty} x^{2n-1} \sum_{1}^{n} (A_r \cos a_r x)\, dx$$

leads us to $$\frac{\cos n\pi}{2n!} \int_0^{\infty} \Sigma(A_r a_r^{2n} \cos x) \frac{dx}{x}$$

with the conditions $\Sigma A_r = 0$; $\Sigma A_r a_r^2 = 0 \ldots \Sigma A_r a_r^{2n} = 0$

which gives finally $$\frac{\cos n\pi}{2n!} \Sigma(A_r a_r^{2n} \log a_r)$$

by Frullani's theorem. (cf. Bromwich, § 170).

Question 867.

(C. KRISHNAMACHARI) :—If the velocity of the Earth is instantaneously altered in the ratio of $\sqrt{2} : 1$, when it is at the extremity of the minor axis of its elliptic orbit, show that the new orbit subsequently described is a parabola whose latus-rectum is twice of the elliptic orbit.

Solution by S. V. Venkatachala Iyer, M. K. Kewalramani, Sadanand, S. Muthukrishnan and S. R. Ranganathan.

The velocity at any point of the orbit is given by

$$v^2=\mu\left(\frac{2}{r}-\frac{1}{a}\right).$$

$\therefore$ At the given instant the velocity is given by

$$v^2=\mu\left\{\frac{2}{a}-\frac{1}{a}\right\}=\frac{\mu}{a}.$$

$\therefore$ v_1^2, the square of the new velocity $=\frac{2\mu}{a}$

$\therefore$ the subsequent motion is given by

$$v^2=\frac{2\mu}{r}$$

which shows that the subsequent path is a parabola.

The semi-latus rectum in the first case is given by $l=\frac{h^2}{\mu}=\frac{v^2p^2}{\mu}$ and in the second by $l_1=\frac{p}{\mu}$. Since p, the perpendicular from the origin to the tangent to the path is the same in both cases

$$l : l_1=v^2 : v_1^2=\frac{\mu}{a} : \frac{2\mu}{a}=1 : 2.$$

Question 888.

(HEMRAJ):—ABCD is a tetrahedron, the edges DA, DB, DC are a, b, c the angles BDC, CDA, ADB, are λ, μ, ν; and 2 X is the sum of the areas of the four faces; and 2 Y the sum of the areas of the four faces less twice the area of the face opposite D; and V the volume. Show that

$$\frac{4X^2Y^2}{9V^2}\Delta\operatorname{cosec}^2\sigma=\Sigma a^2\sin^2(\sigma-\lambda)-\Sigma bc(\cos\lambda-\cos\mu-\nu)$$

where $$2\sigma=\lambda+\mu+\nu,$$

and $$\Delta=1-\Sigma\cos^2\lambda+2\cos\lambda\cos\mu\cos\nu.$$

Solution by H. Br, K. Appukuttan Erady, H. R. Kapadia, K. B. Madhava and S. V. Venkatachala Iyer.

Let $\alpha, \beta, \gamma, \delta$ be twice the area of the faces. Then

$$\alpha = bc \sin \lambda, \text{ and } \beta = ca \sin \mu, \gamma = ab \sin \nu.$$

By considering δ to be the resultant of forces α, β, γ, acting perpendicular to these faces we get

$$\begin{aligned}
\delta^2 &= \alpha^2+\beta^2+\gamma^2+2\Sigma\beta\gamma\cos(\beta\gamma) \\
&= \alpha^2+\beta^2\gamma^2 2\Sigma a^2bc \sin\mu \sin\nu \cos(\beta\gamma) \\
&= \alpha^2+\beta^2+\gamma^2-2abc\Sigma a \cos\lambda-(\cos\mu\cos\nu)] \\
&= (\alpha+\beta+\gamma)^2-2abc\Sigma a\ (\cos\lambda-\cos\overline{\mu+\nu}) \\
&= (\alpha+\beta+\gamma)^2-4abc\Sigma a \sin\sigma \sin(\sigma-\lambda)
\end{aligned}$$

$$\therefore \qquad 4XY = \tfrac{1}{4}[(\alpha+\beta+\gamma)^2-\delta^2] = abc \sin\sigma\Sigma a \sin(\sigma-\lambda)$$

$$\text{or} \qquad \frac{16X^2Y^2}{a^2b^2c^2}\operatorname{cosec}^2\sigma = \Sigma a^2 \sin^2(\sigma-\lambda)+2\Sigma bc \sin(\sigma-\mu)\ \delta \sin(\sigma-\nu)$$

$$= \Sigma a^2\sin^2(\sigma-\lambda)-bc\ (\cos\lambda-\cos\overline{\mu-\nu})$$

From this the given result follows with the help of the well-known formula

$$36V^2 = a^2b^2c^2(1-\cos^2\lambda-\cos^2\mu-\cos^2\nu+2\cos\lambda\cos\mu\cos\nu)$$

Question 915.

(Chas. Saldanha) :—Given $(a+b)$, $(a+c)$ and A, give a geometrical construction for the ΔABC

Solution (1) *by R. Srinivasan,* (2) *by K. J. Sanjana and K. B. Madhava* (3) *by T. Krishna Rao and several others.*

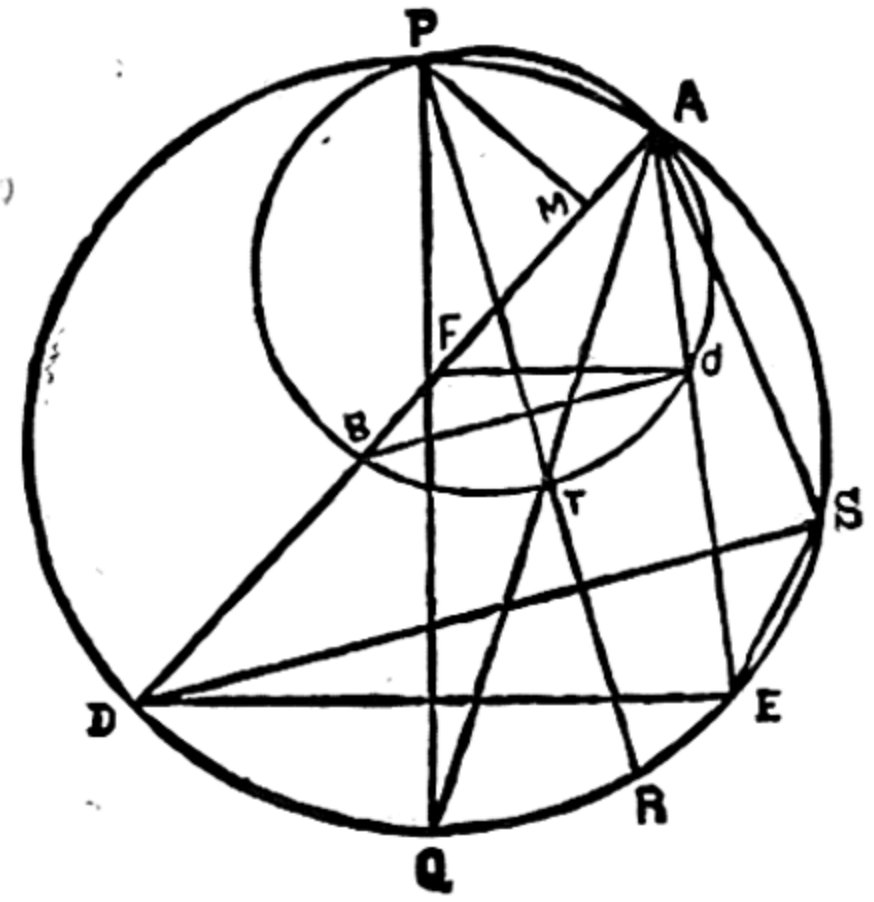
(1) Let ABC be a triangle and I_1 the excentre opposite A. Produce AB, AC to D and E such that BD = BC = CE.

Then Δ DBI $\equiv \Delta$ CBI $\equiv \Delta$ CEI.

$$\therefore\ \hat{DI_1E} = 3\hat{BI_1C} = 3\left(90^\circ-\frac{A}{2}\right).$$

Also I_1 D = I_1 C, ; I E = I B. :—

Hence the following construction.

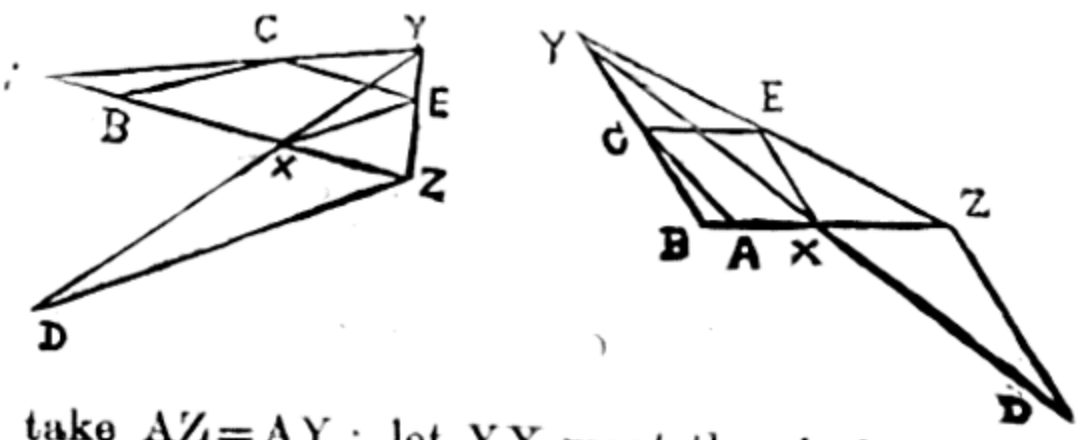


Construct a triangle ADE, such that $AD = a + c$, $AE = a + b$ and $DAE = A$. On DE describe an arc of a circle containing $3\left(90° - \frac{A}{2}\right)$. Let the bisector of the angle A meet this arc in I, with centre I, and radii I D and I E describe circles to cut AE, AD, in CB. Then ABC is the triangle required.

The proof is evident.

(2) More generally let two lengths AX, AY be given inclined at a known angle A; to draw a transversal BC across them so that

$$XB = BC = CY.$$



On AX take $AZ = AY$; let YX meet the circle with centre Z and radius ZA, in D. Join DZ, and draw XE parallel to it to meet YZ; draw EC prallel to AX to meet AY, and CB parallel to EX to meet AX or AX produced.

Then $\frac{CE}{EY} = \frac{AZ}{ZY} = \frac{ZD}{ZY} = \frac{EX}{EY}$, so that $CE = EX$.

∴ CEXB is a rhombus; and $CE = CY$, as $\angle CYE = \angle AZE = \angle CEY$

Thus XB=BC=CY and in the first figure ABC is the triangle required by the question. In the second figure where $AX < \frac{1}{2}AY$, this triangle does not geometrically exist and AB+BX=AX *algebraically.*

(3) Construct a triangle ADE such that AD=$a+c$, AE=$a+b$ and ∠DAE=A. Draw the circum-circle of this triangle and PQ the diameter of the circle perpendicular to DE. Join AQ. Set off a chord QR equal to the difference of the given lengths. Join PR cutting AQ in T. Draw the circle PAT cutting AD, AE in BC respectively. Join BC. Then ABC is the required triangle.

Draw DS parallel BC to meet circle ADE in S and CF parallel to DE to meet AD in F. Join AS, SE. Draw PM perp. to AD.

$\therefore$ PR ⊥ DS and PQ ⊥ DE.

$\angle QPR = \angle EDS$

i.e. chord ES=QR=$c-b$.

Since P is the mid point of segment DAE and PM perpendicular to AD

$\therefore$ $AM = \frac{1}{2}(AD-AE)$

Similarly $AM = \frac{1}{2}(AB-AC)$

i.e. AD−AE=AB−AC or BD=CE.

In the triangles FBC and AES,

the triangles B=ADS=AES

and FCB=SDE=EAS

hence they are similar.

i.e. $$\frac{BC}{FB} = \frac{AE}{ES} = \frac{a+b}{c-b}.$$

Again since FC is parallel to DE

$$\frac{DF}{DA} = \frac{EC}{EA} = \frac{DF-EC}{DA-EA} = \frac{BF}{c-b}.$$

or $$\frac{EC}{BF} = \frac{EA}{c-b} = \frac{a+b}{c-b}.$$

$\therefore$ BC=EC=DB.

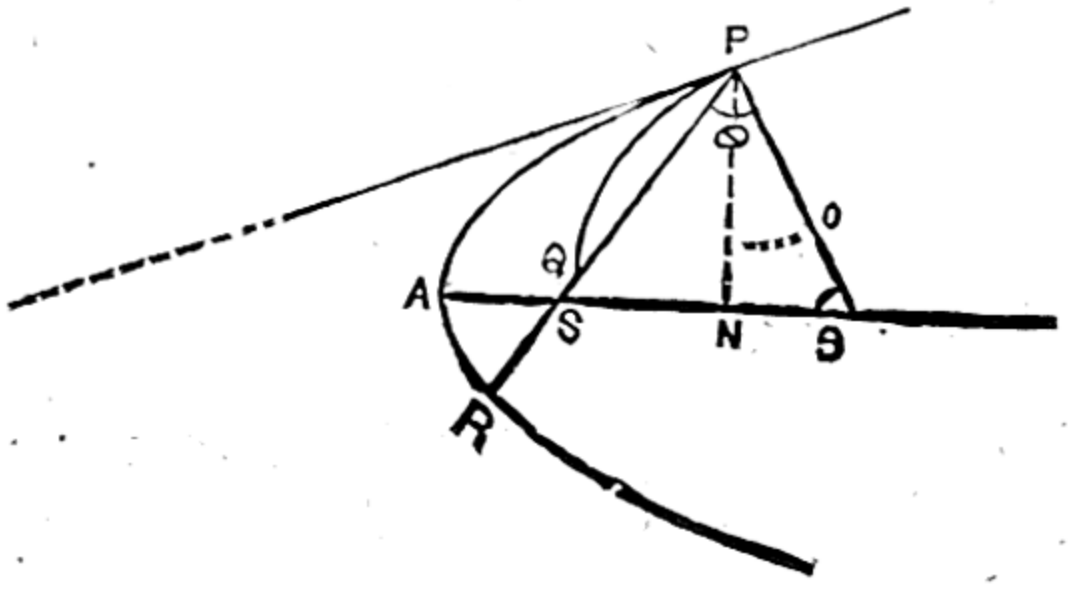
Question 916.

(N. P. Pandya) :—A circle touches a given parabola at P internally and is such that the focal distance of P is divided harmonically by the circle and the parabola. Find the locus of the centre of the circle.

Solution by Hemraj.

Let the equation of the parabola be $y^2=4ax$ and P $(at^2, 2at)$. If O, a point (x, y) on the normal at P, be the centre and P O $=r$, then

$$x=at^2+r\cos\theta,\ y=2at-r\sin\theta \quad \ldots \quad \ldots \quad (1)$$



where SPO $=\theta$ *i.e.*, $\tan\theta=t$.

Since $(PQSR)=-1$, $\frac{2}{PS}=\frac{1}{PQ}+\frac{1}{PR}$ .... (2)

But $PS=a+at^2$, $PQ=2r\cos\theta=2(x-at^2)$ from (1)

$$PR=PS+SR=2a+a\left(t^2+\frac{1}{t^2}\right)$$

∴ On snbstitntion in (2) we have

$$3at^4+2(3a-x)t^2+a-4x=0 \quad \ldots \quad \ldots \quad (3)$$

From (1) $$at^3+(2a-x)t-y=0 \quad \ldots \quad \ldots \quad (4)$$

Eliminating t between (3) O (4) we have the locns of O

$$(4x-a)(x^2-6ax+a^2)^2+y^2(2x^3-78ax^2+90a^2x-18a^3-27ay^2)=0.$$

Question 918.

(C. KRISHNAMACHARY) :—Show that if a and b are both less or greater than unity, and $a<b$, $u_n=\frac{1-a^n}{1-b^n}$ is a decreasing monotonic sequence. Deduce the inequalities,

(1) $\frac{1-b^n}{1-b}.(1-a)>(1-a^n)>\frac{1-b^n}{1-b}\left(\frac{a}{b}\right)^{n-1}(1-a)$, if $o<a<1$

(2) $\frac{b^n-1}{b-1}(a-1)>(a^n-1)>\frac{b^n-1}{b-1}\left(\frac{a}{b}\right)^{n-1}(a-1)$ if $a>1$.

Examine the inequalities for fractional valnes of n.

Solution by K. B. Madhava and S. R. Ranganathan.

In general, if a sequence $\left(\frac{a_n}{b_n}\right)$ is monotonic, the sequence $\left(\frac{\Sigma a_n}{\Sigma b_n}\right)$ is also monotouic and is varying in the same sense.

For, $$\frac{\Sigma a_n}{\Sigma b_n} > \frac{\Sigma a_{n+1}}{\Sigma b_{n+1}}$$

if $$\frac{\Sigma a_n}{\Sigma b_n} > \frac{a_{n+1}}{b_{n+1}}$$

i.e. if $\sum_{r=1}^{n}\left(a_r b_{n+1} - b_r a_{n+1}\right) > 0$ which is true, because each of the term is positive in cosequence of the sequence $\left(\frac{a_n}{b_n}\right)$ being non-increasing

Again if thel sequnce is non-increasing

$$\frac{a_1}{b_1} > \frac{\Sigma a_n}{\Sigma b_n} \text{ for } \sum_{r=2}^{n}\left(a_1 b_r - a_2 b_r\right) > 0.$$

Hence combining $$\frac{a_1}{b_1} > \frac{\Sigma a_n}{\Sigma b_n} > \frac{a_{n+1}}{b_{n+1}}. \quad \ldots \quad \ldots \quad \text{(A)}$$

Take as $a_n = a^{n-1}(1-a)$ or $a^{n-1}(a-1)$ if $o<a<1$ or $a>1$

and $b_n = b^{n-1}(1-b)$ or $b^{n-1}(b-1)$ if $o<b<1$ or $b>1$

and then $a_n = \frac{\Sigma a_n}{\Sigma b_n}$ is a non-increasing monotonic sequence, and the inequalities (A) become (1) and (2) in the problem. (The second involves the condition $a<b$).

The inequalities may be verified for fractional values of n in the usual manner, by multiplying with the denominator, and thus proving it for an integral value of n. The reader may be referred to the examples in p. 21 of Bromwich : *Theory of Infinite Series* for interesting exercises of this type.

Question 920.

(H. R. Kapadia) :—Prove that an integral of

$$\frac{dx}{X^{\frac{1}{2}}} = \frac{dy}{Y^{\frac{1}{2}}}$$

is $$\left(\frac{X^{\frac{1}{2}} + Y^{\frac{1}{2}}}{y-x}\right)^2 = c + e(x+y) + f(x+y)^2$$

where $X = a + bx + cx^2 + ex^3 + fx^4$, and $Y = a + by + cy^2 + ey^3 + fy^4$

Solution by K. J. Sanjana, R. J. Pocock and K. B. Madhava and others.

This is Ex. 2 on p. 241 of Forsyth (Second Edn.).

If we assnme $p=x+y$;

$$\frac{dx}{dt}=\frac{X^{\frac{1}{2}}}{y-x}\ ;\ \frac{dy}{dt}=\frac{Y^{\frac{1}{2}}}{y-x}\ ;$$

the solution proceeds exactly as in Forsyth p 240 with one or two slight changes which are obvions. We obtain finally

$$\left(\frac{dp}{dt}\right)^2=ep+fp^2+c$$

whence substitnting for $\frac{dp}{dt}$ we get

$$\left[\frac{X^{\frac{1}{2}}+Y^{\frac{1}{2}}}{y-x}\right]^2=C+e(x+y)+f(x+y)^2$$

[*Vide* also, Boole : *Differential Equations.*, Ch. VI ; Roberts : *Integral Calculus* § § 87, 88 ; Dixon : *Elliptic Functions*, § 40.]

Question 923.

(M. K. Kewalramani) :—Supposing $\phi(x)$ vanishes when $x=0$, ⊙ut neither it nor its differential coefficients vanish for any other valneofb$x$ prove the following identity :—

$$x\phi(x)-\frac{x^2}{2!}\phi'(x)+\frac{x^3}{3!}\phi''(x)-\frac{x^4}{4!}\phi'''(x)+\ldots$$
$$\equiv A_1x\phi(x)+A_3x^3\phi''(x)+A_5x^5\phi''''(x)+\ldots$$

where

$$A_{2n+1}=2\frac{(2^{2n}-1)}{(2n)!}\ B_{2n-1},\ (B_{2n-1}\text{ being the }n^{th}\text{ Bernoullian.})$$

Also, show that

$$\frac{\pi}{2}\ A_1-\frac{\pi^3}{2^3}\ A_3+\frac{\pi^5}{2^5}\ A_5-\ldots=1$$

Remarks by Sadanand.

The identity is not true as it stands. We are required to prove that

$$\left(1-e^{xD}\right)\equiv\tan\left(^1xD\right)$$

where D stands for the operative symbol $\frac{d}{dx}$, which is impossible.

The series

$$\frac{\pi}{2}\ A_1-\frac{\pi^3}{2^3}\ A_3+\ldots=-i\tan\left(i\frac{\pi}{4}\right)\neq 1;$$

but the series

$$\frac{\pi}{2}\ A_1+\frac{\pi^3}{2^3}\ A_3+\ldots=\tan\frac{\pi}{4}=1.$$

QUESTIONS FOR SOLUTION.

[N.B.—Question 965 by Mr. A. C. L. Wilkinson which appeared in June 1918 should be numberedas 965 A.]

969. (H. Br.) :—From the vertex D of a tetrahedron ABCD straight lines DL,DM,DN, are drawn parallel and proportional to the perpendiculars from a point P onthe faces BCD, CAD, ABD, respectively. Q is the isogonal conjugate of P with respect to ABCD. Show that PD passes through the circumcentre of DLMN, and QD is perpendicular to LMN. Show further that if DL, DM, DN. are twice the lengths of the three perpendiculars, but are drawn in the opposite direction in each case, P is the circumcentre of DLMN.

970. (S. R. RANGANATHAN) :—Show that

$$x^n \prod_{r=1}^{\infty}\left(1+\frac{x^{2n}}{r^{2n}}\right) = \frac{1}{2^{n/2}\pi^n} \prod_{r=1}^{n-1} (\cosh 2\pi\alpha_r x - \cos 2\pi\beta_r x)$$

when n is even,

$$= \frac{1}{2\frac{1}{2}(n-r)\pi^n} \sinh \pi x \prod_{1}^{n-2} (\cosh 2\pi\alpha_r x - \cos 2\pi\beta_r x)$$

when n is odd,

where $\alpha_r = \sin\frac{r\pi}{2n}$, and $\beta_r = \cos\frac{r\pi}{2n}$, r being an odd integer.

971. (A. A. KRISHNASWAMI AIYANGAR M.A , L.T.) :—A rigid spherical shell of radius a contains a mass M of gas in which the pressure is K times the density, and the gas is repelled from a fixed external point O (distant c from the centre) with a force per unit of mass equal to K times the distance. Prove that the resultant pressure of the gas on the shell is

$$\frac{KM}{c}\cdot\frac{5c^2-a^2}{5c^2+a^2}.$$ (Besant and Ramsay) *Hydrostatecs.*

972. (K. B. MADHAVA) ;—Solve.

$$\delta(x)+\delta(y)+\delta(z)=B.$$

$$0=[\delta(y)-\delta(x)][\delta'(y)+\delta'(z)]+[\delta(y)-\delta(z)][\delta'(y)+\delta'(x)]$$

$$0=[\delta(x)-\delta(z)][\delta'(x)+\delta'(y)]+[\delta(x)-\delta(y)][\delta'(x)+\delta'(z)]$$

where δ is the Weiertrassian Elliptic function and δ', its derivate.

[This is needed in the solution of Lame's equation when $n=3$;

$$\frac{1}{w}\frac{w^2 d}{dr^2} = 12\,\delta\,(z)+5B.].$$

973. (A. C. L. WILKINSON) :—Prove that

$$\mathrm{Dsn}\,(u_1+u_2+u_3) = \begin{vmatrix} s_1, & s_1^1, & s_1^2 \\ s_2, & s_2^1, & s_2^2 \\ s_3, & s_3^1, & s_3^2 \end{vmatrix};\ \mathrm{Dcn}\,(u_1+u_2+u_3) = -\begin{vmatrix} c_1, & c_1^1, & c_1^2 \\ c_2, & c_2^1, & c_2^2 \\ c_3, & c_3^1, & c_3^2 \end{vmatrix};$$

$$k^4\,\mathrm{D}\,\mathrm{dn}(u_1+u_2+u_3) = -\begin{vmatrix} d_1, & d_1^1, & d_1^2 \\ d_2, & d_2^1, & d_2^2 \\ d_3, & d_3^1, & d_3^2 \end{vmatrix};$$

where $\mathrm{D}=\begin{vmatrix} 1, & s_1^2, & s_1c_1d_1 \\ 1, & s_2^2, & s_2c_2d_2 \\ 1, & s_3^2, & s_3c_3d_3 \end{vmatrix}$ and $s_1=\mathrm{sn}u_1$, $s_1'=\dfrac{d}{du_1}(\mathrm{sn}u_1)=\mathrm{cn}u_1\,\mathrm{dn}u_1$ etc.

Prove that

$$\mathrm{E}(u_1)+\mathrm{E}(u_2)+\mathrm{E}(u_3)-2\mathrm{E}(x)$$
$$=\frac{k^2d_1d_2d_3d_4}{k^{12}+d_1d_2d_3d_4}\left(\frac{s_1c_1}{d_1}+\frac{s_2c_2}{d_2}+\frac{s_3c_3}{d_3}+\frac{s_4c_4}{d_4}\right)$$
$$-\frac{2k^2\mathrm{sn}x\,\mathrm{cn}x.\mathrm{dn}(x-u_3-u_4)\,\mathrm{dn}\,(x-u_2-u_4)\,\mathrm{dn}\,(x-u_2-u_3)}{k^{12}+\mathrm{dn}\,x\,\mathrm{dn}\,(x-u_3-u_4)\,\mathrm{dn}\,(x-u_2-u_4)\,\mathrm{dn}\,(x-u_2-u_3)}$$

where $x=\frac{1}{2}(u_1+u_2+u_3+u_4)$ and s_1,c_1,d_1 are written as usual for $\mathrm{sn}u_1$, $\mathrm{cn}u_1$, $\mathrm{dn}u_1$.

974. (A. C. L. Wilkinsón) :—If two confocal conics are such that triangles ABC can be inscribed in the one which are circumscribed to the other and if D, E, F are the points of contact, prove that (1) the area of the triangle DEF is proportional to the product of the diameters conjugate to D, E, F: (2) the ratio of the areas of the triangles ABC. DEF is constant: (3) the loci of the points of concurrence of the normals at A, B, C and D, E, F are ellipses (4) the line joining the points of concurrence of the normals at A, B, C and D, E, F is a normal to an ellipse: (5) if from the points of concurrence the fourth normals be drawn to the two confocals the line joining their feet is also normal to an ellipse.

(Suggested by Question 947 which shows that in all triangles ABC the ratio R : r is constant.)

975. (Professor K. J. Sanjana):—A uniform right pyramidal solid on a square base is divided into two halves by a plane through the vertex and one diagonal of the base, and the two parts are hinged together at the vertex. If the solid floats in a liquid which is such that the plane of flotation is a square whose side bears to the side of the base the ratio $r:1$, prove that the parts will not separate if $\sec\theta \geq r^{\frac{1}{2}}$ when the vertex is within the fluid, and if $\mathrm{cosec}\theta \leq (3r^3-r^2-r)^{\frac{1}{2}}$ when the vertex is without, $-\theta$ being the dihedral angle between the planes of the base and a slant side of the pyramid.

976. (K. J. Sanjana, M. A.):—Find the probability that the roots of (i) the biquadratic $ax^4+bx^2+c=0$, (ii) the cubic $ax^3+bx+c=0$, may be all real, where the numbers a, b, c are taken at random.

977 (M. K. Kewalramani):—In the series

$$\frac{1}{3}-\frac{1}{7}+\frac{1}{11}-\frac{1}{15}+\frac{1}{19} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \infty$$

if the terms are deranged in such a manner that to every 6561 positive terms correspond 256 negative terms, the terms in each group having their original order, show that the sum of the new series is

$$\frac{\sqrt{2}}{8}\left[\pi-\log\frac{\sqrt{2}+1}{\sqrt{2}-1}\right]+\log\frac{3}{2}$$

978. M. K. KEWALRAMANI):—If two circles are such that quadrilaterals can be inscribed in the one which are circumscribed to the other, prove that the ratio of the sum of the reciprocals of the diagonals of any two quadrilaterals is constant.

[Suggested by Q. 963]

979. (F. H. V. GULASEKHARAM):—If $u_1, u_2, u_3, u_4, u_5, u_6$ are variable arguments connected by the relations $u_2-u_1=u_3-u_2=u_4-u_3=u_5-u_4=u_6-\varepsilon=\frac{K}{3}$

and if d_r stands for $dn\ u_r$, shew that

(1) $(d_1d_2+d_2d_3+d_3d_4+d_4d_5+d_5d_6+d_6d_1)$ is constant.

(2) $(d_1+d_4)(d_2+d_5)+(d_2+d_5)(d_3+d_6)+(d_3+d_6)(d_1+d_4)$ is constant.

In each case determine the constant in terms of the modulus k and the sn, cn, dn, functions of $\frac{K}{3}$.

980. (F. H. V. GULASEKHARAM):—A triangle Δ is inscribed in a given conic $S=0$ and circumscribed to a given confocal conic $S_1=0$. A triangle Δ_1 is inscribed in the conic $S_1=0$ with its sides parallel to those of Δ. Shew that

(1) the circum-radius of the triangle Δ_1 = the in radius of the triangle Δ;

(2) the locus of the circum-centre of the triangle Δ_1 for all positions of Δ is another fixed confocal conic.

981. (R. VYTHYNATHASWAMY). $A_1\ A_2 \ldots A_n$ are n unequal real numbers If $L_{pq}\ldots$ and $\ldots G_{pq}$ denote respectively the least and greatest of the numbers A_p, A_q....., prove that

$$L_{123\ldots n}=\Sigma G_1-\Sigma G_{12}+\Sigma G_{123}-\ldots$$

$$G_{123\ldots n}=\Sigma L_1-\Sigma L_{12}+\Sigma L_{123}-$$

Use these results to express the G. C. M (L. C. M.) of n positive integers as a function of the integers and their mutual L.C M's (G.C.M's).

982. (R. VYTHYNATHASWAMY):—If I and SIS are both involutions, prove that S must be an involution unless I contains the fixed points of S

983. (R. VYTHYNATHASWAMY):—If Ω and S Ω are both periodic and of the same period, shew that thsir fixed points make with the fixed points of S ranges of equal cross ratio.

List of Periodicals Received.

(*From 16th May to* 15*th July* 1918).

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1. American Mathematical Monthly, Vol. 25, No 3, March 1918.

2. Astrophysical Journal, Vol. 47, Nos. 1 and 2, January and March 1918.

3. Bulletin of the American Mathematical Society, Vol. 24 No. 7, April 1918.

4. L'Intermediaire der Mathematiciens, Vol. 25, Nos. 3, 4, 5 and 6, March, April, May and June 1918

5. Mathematical Questions and Solutions, Vol. 5, Nos. 1, 2 & 3, Jan. Feb. and March. 1918. (5 Copies).

6. Mathematics Teacher, Vol. 10, No. 3, March 1918.

7. Monthly Notices of the Royal Astronomical Society, Vol. 78, Nos. 3 & 4, January and February 1918.

8. Philosophical Magazine, Vol. 35, No. 209 May 1918.

9. Popular Astronomy Vol. 26, Nos. 3, 4, & 5, March, April and May 1918 (3 Copies).

10. Proceedings of the London Mathematical Society, Vol. 16, No. 1913.

11. Proceedings of the Royal Society of London, Vol. 94, Nos. 6 April and May 1918.

12. School Science and Mathematics, Vol. 18, Nos. 4 and 4, March 1918. (2 Copies).

13. Tohoku Mathematical Journal, Vol 18, No. 3, April 1918.

A Note in Combinatory Analysis.

By R. Vythynathaswamy.

§ 1. The following lemma will be of use in the course of the present paper.

Lemma :

The number of ways of placing λ distinct things in the first of two rows of r vacant places and μ distinct things in the second, where $\lambda \not> r$, and $\mu \not> r$, and each of the λ as well as each of the μ things should be used at least once, so as to satisfy the condition that if k_1, l_1, two members of the first row, contain the same λ-thing, k_2, l_2, the corresponding members in the second row should not contain the same μ-thing, is the coefficient of x^r in

$$\left[\Delta_1^{\lambda}\,\Delta_2^{\mu}(1+x)^{t_1 t_2}\right]_{t_1=t_2=0}$$

where the operators Δ_1 and Δ_2 refer to t_1 and t_2 respectively.

Proof. Let the λ things be $\alpha_1, \alpha_2, \ldots \alpha_\lambda$ and the μ things $\beta_1, \beta_2, \ldots \beta\mu$.

Fill the first row with k_1. α_1's, k_2. α_2's $\ldots k_\lambda$ α_λ's where $k_n (n=1,2,\ldots\lambda)$ is a positive non-zero integer and $\Sigma k_n = r$.

The corresponding number of ways of filling the second row will then be

$$\mu C_{k_1} \cdot \mu C_{k_2} \cdot \mu C_{k_3} \cdots \mu C_{k_\lambda}$$

Hence the total number of ways would be

$$\sum_k \mu C_{k_1}\, \mu C_{k_2} \cdots \mu C_{k_\lambda} \text{ where } \Sigma k_n = r \text{ and } k_n \neq 0$$

and this is the coefficient of x^r in $[(1+x)^\mu - 1]^\lambda$.

If now the condition that each of the βs should occur once at least be added, the total number of ways is

$$[(1+x)^\mu - 1]^\lambda - \mu c_1 [(1+x)^{\mu-1} - 1]^\lambda + \mu c_2 [(1+x)^{\mu-2} - 1]^\lambda - \ldots$$

which is easily seen to be a symmetric function of λ and μ and equal to

$$\left[\Delta_1^{\lambda}\,\Delta_2^{\mu}(1+x)^{t_1\, t_2}\right]_{t_1=t_2=0}.$$

This we shall write also as $\Delta_1^{\lambda}\Delta_2^{\mu}(1+x)^{0\cdot 0}$.

We may also adopt a different type of argument which is perfectly general.

If there are λ distinct things of one kind and μ distinct things of another kind, then the number of ways of forming r distinct combina-

tions of a λ and a μ thing is evidently $\lambda\mu^{C}r$, and these correspond to the number of ways of arranging the λ and μ things in two rows of r spaces each, with two dissimilar λ—things always opposite to two similar μ—things, without, however, the assurance that each μ—thing or each λ—thing occurs at least once. Therefore, if we add the condition that each λ—thing is to occur at least once, the number of ways is

$$\lambda\mu^{C}r-\lambda^{C}1(\lambda-1)\mu^{C}r+\lambda^{C}2(\lambda-2)\mu^{C}r-\ldots=\Delta^{\lambda}0\cdot\mu^{C}r$$

(cf. Whitworth: *Choice and Chance*

Adding the further condition that each μ—thing should occur also once at least, we see that the total number of ways is

$$\Delta_1^{\lambda}\Delta_2\mu_{0\cdot0}{}^{C}r$$

which agrees with the previous result.

Corollary. More generally, we shall also have:

The number of ways of placing $\lambda_1,\lambda_2\ldots\lambda_n$ things respectively in n rows of r spaces each, so that each thing occurs at least once, and no two columns are identical, is equal to

$$\Delta_1^{\lambda_1}\Delta_2^{\lambda_2}\ldots\Delta_n^{\lambda_n}0\,0\cdot0\ldots{}^{C}r$$

§ 2. We can now find *the number of ways in which a set of r different factors of N can be chosen so as to have the greatest common measure unity.*

We will denote this by $\phi_r(N)$ and seek to express $\phi_r(pq)$ where p and q are mutually prime in terms of $\phi_k(p)$ and $\phi_k(q)$. If p and q are mutually prime, every factor of pq is uniquely resoluble into a factor of p and a factor of q. Thus every factor is specified by a factor of p and a factor of q. To specify r distinct factors of pq, then we have to take λ factors of p and μ factors of q where, of course, neither λ nor μ is to be greater than r, and fill with them the two rows of r vacant places satisfying the conditions of the Lemma; that is

$$\phi_r(pq)=\text{coefficient of } x^r \text{ in } \Sigma\Sigma\phi_\lambda(p)\phi_\mu(q)\Delta_1^{\lambda}\Delta_2^{\mu}(1+x)^{0.0} \qquad (\lambda\not>r,\ \mu\not>r.$$

$$=\Sigma\Sigma\phi_\lambda(p)\phi_\mu(q)\Delta_1^{\lambda}\Delta_2^{\mu}\,0.0^{C}r$$

Similarly, if $a_1, a_2,\ldots a_n$ are mutually prime, we shall have

$$\phi_r(a_1a_2\ldots a_n)=\Sigma\phi_{\lambda_1}(a_1)\phi_{\lambda_2}(a_2)\ldots\phi_{\lambda_n}(a_n)\Delta_1^{\lambda_1}\Delta_2^{\lambda_2}\ldots\Delta_n^{\lambda_n}000\ldots{}^{C}r$$

We have also the result, that if p is a prime $\qquad [\lambda_k\not>r\ k=1,2\ldots n.]$

$$\phi_r(p^{\alpha}) = {}^{\alpha}C_r - 1 ;$$

and therefore, that if $N = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_n^{\alpha_n}$

$$\phi_r(N) = \Sigma \phi_{\lambda_1}(p_1^{\alpha_1}) \ldots \phi_{\lambda_n}(p_n^{\alpha_n}) . \quad \Delta_1^{\lambda_1} \ldots \Delta_n^{\lambda_n} 000 \ldots C_r$$

$$= \Sigma \alpha_1 {}^{C}_{\lambda_1 - 1} \cdots \alpha_n {}^{C}_{\lambda_n - 1} . \Delta_1^{\lambda_1} \ldots \Delta_n^{\lambda_n} \; 000 {}^{C}r$$

$$= (1 + \Delta_1)^{\alpha_1} \ldots (1 + \Delta_n)^{\alpha_n} \Delta_1 \Delta_2 \ldots \Delta_n 000 {}^{C}r$$

$$= \Delta_1 \Delta_2 \ldots \Delta_n \left[\alpha_1 \alpha_2 \ldots \alpha_n {}^{C}r \right].$$

Thus if $N = 120$ and $r = 2$, $\phi_r(N) = 17$.

§ 3. If $\phi_r'(N)$ be defined as the number of sets of r factors of N (not including N itself) which can be found so as to have the greatest common measure unity, we evidently have

$$\phi_r' + \phi'_{r-1} = \phi_r \text{ (for values of } r \text{ up to 2)}$$

and we shall also suppose $\phi_1' = \phi_1 = 1$. We then have

$$\phi'_r = \phi_r - \phi_{r-1} + \phi_{r-2} - \ldots$$

$$= \Delta_1 \Delta_2 \ldots \Delta_n [\alpha_1 \alpha_2 \ldots \alpha_n C_r - \alpha_1 \alpha_2 \ldots \alpha_n C_{r-1} + \ldots]$$

$$= \Delta_1 \Delta_2 \ldots \Delta_n \left[\alpha_1 \alpha_2 \ldots \alpha_n - 1 {}^{C}r \right]$$

We therefore have

$$\phi_1' - \phi'_2 + \phi'_3 - \ldots = \Delta_1 \Delta_2 \ldots \Delta_n \left[\alpha_1 \alpha_2 \ldots \alpha_n - 1 {}^{C}_1 - \alpha_1 \alpha_2 \ldots \alpha_n - 1 {}^{C}_2 + \ldots \right]$$

$$= - \Delta_1 \Delta_2 \ldots \Delta_n (1-1)^{\alpha_1 \alpha_2 \ldots \alpha_n - 1}$$

from which by writing $E_r - 1$ for Δ_r and expanding, we see that the terms in which the index is not zero will vanish. If any one of the αs is different from unity, the index of none of the terms in the expanded series vanishes. Hence *when* N *contains repeated prime factors the series identically vanishes.* If on the other hand all the αs are unity, there is one term for which the index is zero; *viz* $(-)^{n+1}(1-1)^0 = (-)^{n-1}$, and therefore, *if* N *contains* n *prime factors none being repeated the series equals* $(-)^{n-1}$.

§ 4. We may notice that if $\psi_r(N)$ denotes the number of sets of r factors of N which can be chosen so as to have the least common multiple N, then $\psi_r(N) = \phi_r(N)$. For if we divide N by a set of factors having the least common multiple N, we get a set of factors having the greatest common factor unity, Similarly also we find $\psi_r'(N) = \phi_r'(N)$ where $\psi_r'(N)$ is the same as $\psi_r(N)$ with unity excluded.

This method of reasoning may be employed to obtain two familiar results.

Let $t_1, t_2 \ldots$ be the factors of N, N being excluded. If $\phi(N)$ be the number of factors of N we have,

$$\phi(N)-1=\Sigma\phi(t_1)-\Sigma\phi(t_{12})+\Sigma\phi(t_{123})\ldots$$

where t_{12-r} is the G. C. M. of $t_1, t_2, \ldots t_r$ and is thus itself one of the ts

that is, $$\phi(N)=1+\Sigma\phi(t_k)\left[1-\phi_2'\left(\frac{N}{t_k}\right)+\phi_3'\left(\frac{N}{t_k}\right)-\ldots\right]$$

$$=1+\Sigma\phi\left(\frac{N}{p_1}\right)-\Sigma\phi\left(\frac{N}{p_1p_2}\right)+\ldots$$

which is a familiar result. (Cf. Chrystal: *Algebra*, Vol. II.)

Similarly, the numbers less than N and *not* prime to it will be multiples of one or more of the factors $t_1, t_2 \ldots t_n$ and will therefore be $\sum\frac{N}{t_k}$ in all, which ennmeration, however, is not simplex. The simplex enumeration is $\Sigma\frac{N}{t_1}-\Sigma\frac{N}{t_{12}}+\Sigma\frac{N}{t_{123}}-\ldots$

where $t_{12\ldots r}$ is the least common multiple of $t_1, t_2, \ldots t_r$.

This last series is by the definition of ψ'

$$=\Sigma\frac{N}{t_k}[1-\psi_2'(t_k)+\psi_3'(t_k)-\ldots]$$

and the expression, in the square brackets, is either 0 or $(-1)^{n-r}$ according as t_k contains repeated prime factors, or n non-repeated prime factors. Hence, finally, the number of numbers less than N and not prime to it is

$$\sum\frac{N}{p_1}-\sum\frac{N}{p_1p_2}+\sum\frac{N}{p_1p_2p_3}.$$

which is a well known result. (Cf. Smith: *Algebra*, p. 488).

Postscript: Extension of Fermat's Theorem.

It is well known that a rational transformation of the nth degree has $n+1$ fixed points and the mth power of the transformation which is of degree n^m has n^m+1 fixed points. But if t be any factor of m, all the fixed points of S^t (where S is the rational transformation in question) are also fixed points of S^m. Hence, if $t_1\, t_2 \ldots t_k$ be the factors of m, m itself being excluded, the number of *special* fixed points of S^m would be equal to

$$(n^m+1)-\Sigma(n^t{}_1+1)+\Sigma(n^t{}_{12}+1)\Sigma(n^t{}_{123}+1)$$

where $t_{12\ldots r}$ is the G.C.M of $t_1\, t_2 \ldots t_r$.

This series is equal to $n^m - \Sigma n^{\frac{m}{p_1}} + \Sigma n^{\frac{m}{p_1 p_2}} - \text{etc.}$

where $p_1\ p_2 \ldots$ are the prime factors of m.

We shall shew now that if p is a special fixed point of S^m, then $S(p)$ is also a special fixed point.

It is evident of course that $S(p)$ is a fixed point of S^m. To shew that it is a special fixed point, assume it to be a fixed point of S^t where t is a factor of m. Then we have the following series

p, $S(p)$, $S(S(p)), \ldots S^{t-1}(S(p))$, $S^t(S(p))\ \{=S(p)\}$, etc.

It is evident from this scheme that $S^m(p) = S^{t-1}(S(p)) = S^t(p)$

Since $S^m(p) = p$, this shews that p also is a fixed point of S^t. This however is contrary to the hypothesis that p is a special fixed point. Hence $S(p)$ is a special fixed point. Hence if p_1 is one special fixed point, p_1, $S(p_1)$, $S^2(p_1) \ldots S^{m-1}(p_1)$ are all special fixed points *and are all distinct*. In fact, if $S^r(p_1) = S^n(p_1)$, then $S^r(p)$ is a fixed point of S^{n-r}; which is impossible since $S^r(p)$ is a special fixed point. Again if p_2 be a special fixed point not included in the above series, then p_2, $S(p_2)$, $S^2(p) \ldots S^{m-1}(p_2)$, are all special fixed points and are all distinct both from one another and from the points of the previous series. In fact, the assumption that $S^r(p_1) = S^n(p_2)$, leads to $S^{m+r-n}(p_1) = p_2$, which contradicts the hypothesis that p_2 is not included in the first series.

Proceeding in this manner we finally come to the conclusion, that the special fixed point can be distributed into sets of cyclic m—groups. Hence.

If n and m be any two integers and $p_1\ p_2\ p_3 \ldots$ are the prime factors of m, then the series.

$$n^m - \Sigma n^{\frac{m}{p_1}} + \Sigma n^{\frac{m}{p_1 p_2}} - \Sigma n^{\frac{m}{p_1 p_2 p_3}} +$$

is exactly divisible by m.

If m is a prime, this reduces to Fermat's Theorem.

It will be noticed that in this extension of Fermat's theorem no restrictions are imposed on either of the two numbers n, m.

Extension of M'cay's Theorem.

By S. Narayanan, B.A., L T., Madura College.

M'Cay's extension of Feuerbach's theorem is '*If the axis major of an inscribed conic passes through the circumcentre, the auxiliary circle touches the nine-points circle.*' It is proposed, in this paper, to shew generally the relation between the auxiliary circle of *any* inconic whose axis major passes through a *given point* P and the pedal circle of P. The main theorem of this paper may be enunciated as follows:

If the axis major of an inscribed conic of a triangle passes through a given point P, the auxiliary circle of this conic and the pedal circle of P *are coaxal with the director circle of the inconic touching the axis major at P, and the common radical axis envelopes the orthopolar conic of P.*

Lemma I. If $x(y^2-z^2)=l,\ y(z^2-x^2)=m,\ z(x^2-y^2)=n,$

then $$\frac{x(y^2+z^2)}{l(m^2+n^2-l^2)}=\frac{y(z^2+x^2)}{m(n^2+l^2-m^2)}=\frac{z(x^2+y^2)}{n(l^2+m^2-n^2)}=\frac{-xyz}{lmn}.$$

This follows immediately on reduction.

Lemma II. If a, b c, denote the circles described on the sides BC, C A AB of a triangle ABO as diameters, then the director circle of any inscribed conic of ABC may be written in the form $\lambda a+\mu b+\nu c=0$.

This is also obvious by rearranging the usual equation of the director circle of an inconic in terms of the varying coefficients of the conic.

Lemma III. The tricyclic equation $al^2\alpha'+bm^2\beta'+cn^2\gamma'=0,$

represents the director circle of the inscribed conic which touches $l\alpha+m\beta+n\gamma=0$ at α', β', γ'. [Here *a, b, c* denote, as before, the circles on the sides as diameters, *l, m, n* are variables subject to the condition $l\alpha'+m\beta'+n\gamma'=0$ and α', β', γ' are the normal coordinates of any point].

This follows immediately from Lemma II.

Lemma IV. If P be any point in the plane of a triangle ABC, and if AP, BP, CP meet the opposite sides in D, E, F and if K, L, M be the orthopoles of AP, BP, CP respectively, then the locus of the orthopoles of straight lines passing through P, which is called the orthopolar conic of P, is the conic passing through K, L, M and touching the common chords of the pedal circle of P and the circles on AP, BP, CP as diameters.

Let $l\alpha+m\beta+n\gamma=0$ be any straight line passing through α',β',γ', the normal coordinates of the orthopole $l\alpha+m\beta+n\gamma=0$ are given by

$$\left.\begin{aligned}\alpha&=(m\cos B+n\cos C-l\cos B\cos C)\,(m\cos C+n\cos B-l)\\ \beta&=(n\cos C+l\cos A-m\cos C\cos A)\,(n\cos A+l\cos C-m)\\ \gamma&=(l\cos A+m\cos B-n\cos A\cos B)\,(l\cos B+m\cos A-n)\end{aligned}\right\}\ (I)$$

Now, by means of $l\alpha'+m\beta'+n\gamma'=0$, substitute for l in terms of m and n in (I), we have thus three equations in m^2, mn, n^2. And again eliminating m and n between these three equations, the result evidently assumes the form $QR=P^2$ ... ... (A)

[The details of working are omitted as being too tedious for reproduction here].

Similarly substitute for m in terms of n and l in (I) and eliminate n and l between the three equations. The result will be of the form $Q_1R_1=P_1^2$ ... ... (B)

Again doing the same thing for n we have $Q_2R_2=P_2^2$ ... (C)

On a simple inspection of the results (A), (B), (C), it will be found that the orthopolar conic in question is *circumscribed* to the triangle formed by

$$\begin{aligned}\alpha&=(\beta'\cos B-\gamma'\cos c)\,(\beta'\cos c-\gamma'\cos B),\\ \beta&=\cos c\,(\beta'+\gamma'\cos A)\,(\beta'\cos A+\gamma'),\\ \gamma&=\cos B\,(\beta'+\gamma'\cos A)\,(\beta'\cos A+\gamma'),\end{aligned}$$

and two other points whose coordinates can easily be written down from cyclic symmetry.

The same is *inscribed* in the triangle formed by

$$\begin{aligned}\alpha&=-\{\alpha'^2(\cos^2 B+\cos^2 C)+\alpha'\beta'\cos C\,(1+\cos^2 B)\\ &\qquad+\alpha'\gamma'\cos B\,(1+\cos^2 C)+2\beta'\gamma'\cos B\cos C\},\\ \beta&=\alpha'^2\cos C\,(1+\cos^2 A)+\alpha'\beta'\,(\cos^2 A+\cos^2 C)\\ &\qquad-\alpha'\gamma'\cos A\,(1+\cos^2 C)-2\beta'\gamma'\cos C\cos A,\\ \gamma&=\alpha'^2\cos B\,(1+\cos^2 A)-\alpha'\beta'\cos A\,(1+\cos^2 B)\\ &\qquad+\alpha'\gamma'\,(\cos^2 A+\cos^2 B)-2\beta'\gamma'\cos A\cos B,\end{aligned}$$

and two other points whose coordinates can easily be written down from cyclic symmetry.

The three former set of points can easily be recognized to be the orthopoles K, L, M of AP, BP, CP respectively. And the joins of the three latter set of points are the chords of intersection of the pedal circle of P with the circles on AP, BP, CP as diameters.

Let us now consider the theorem proper.

Let x,y,z be the normal coordinates of the focus of an inconic whose axis major passes through $P(\alpha',\beta',\gamma')$.

The equation of the major axis is

$$\alpha x(y^2-z^2)+\beta y(z^2-x^2)+\gamma z(x^2-y^2)=0.$$

If this passes through α',β',γ', we have

$$\alpha' x(y^2-z^2)+\beta' y(z^2-x^2)+\gamma' z(x^2-y^2)=0. \quad \ldots \quad \ldots \quad (D)$$

The equation of the auxiliary circle, *viz.* the pedal circle, of (x,y,z) is

$\Sigma\beta\gamma \sin A.\ \Sigma yz \sin A.\ \Sigma x \sin A.$

$=\Sigma\alpha \sin A.\ \Sigma\alpha x \sin B \sin C\ (y+z\cos A)\ (y\cos A+z)$

Arranging this in terms of x,y,z and powers, we have

$$ax(y^2+z^2)+by(z^2+x^2)+cz(x^2+y^2)+k\,xyz=0 \ldots \quad \ldots \quad (E)$$

where a, b, c denote the circles on BC, CA, AB as diameters

i.e.

$$a \equiv \beta\gamma+\gamma\alpha\cos C+\alpha\beta\cos B-\alpha^2\cos A$$
$$b \equiv \beta\gamma\cos C+\gamma\alpha+\alpha\beta\cos A-\beta^2\cos B$$
$$c \equiv \beta\gamma\cos B+\gamma\alpha\cos A+\alpha\beta-\gamma^2\cos C$$

and

$$k \equiv 2\Sigma\beta\gamma\cos B\cos C-\Sigma\alpha^2(1+\cos^2 A).$$

Now let $x(y^2-z^2)=l$, $y(z^2-x^2)=m$, $z(x^2-y^2)=n$, the relation (D) assumes the form

$$l\alpha'+m\beta'+n\gamma'=0 \quad \ldots \quad \ldots \quad \ldots \quad (F)$$

and with the aid of Lemma I, the relation (E) becomes

$$al\,(m^2+n^2-l^2)+bm\,(n^2+l^2-m^2)+cn\,(l^2+m^2-n^2)=k\,l\,m\,n. \quad (G)$$

Now (G) is the auxiliary circle of the inconic whose axis major is

$$l\alpha+m\beta+n\gamma=0.$$

Transforming (G) with the aid of (F), we have

$$\Sigma\, a\, l^2\,(m\gamma'.\sqrt{\alpha'^2-\beta'^2}+n\beta'.\sqrt{\alpha'^2-\gamma'^2})$$
$$+l\,m\,n\,\{k\alpha'\beta'\gamma'+\Sigma\, a\,\alpha'\,(\beta'^2+\gamma'^2)\}=0 \quad \ldots \quad (H)$$

The co-efficient of $l\,m\,n$ in(H) can easily be identified with the pedal circle of α', β', γ'. Hence, denoting this by Θ, the auxiliary circle becomes after further reduction with the aid of $l\alpha+'m\beta'+n\gamma'=0$,

$$l\,m\,n\,\alpha'\,\Theta+(m\gamma'.\sqrt{\alpha'^2-\beta'^2}+n\beta'.\sqrt{\alpha'^2-\gamma'^2})\,(a\,l^2\,\alpha'+b\,m^2\beta'+cn^2\gamma')=0.$$

Here let $a\,l^2\,\alpha'+bm^2\beta'+cn^2\gamma'=0$ be denoted by ϕ and $m\,\gamma'\,(\alpha'^2-\beta'^2)$ $+n\,\beta'\,(\alpha'^2-\gamma'^2)\equiv\lambda$ the auxiliary circle may be written in the form

$$lmn\,\alpha'\Theta+\lambda\phi-0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (J)$$

Now the relation J shews that the circles ϕ and Θ are coaxal with the auxiliary circle. And Lemmas II and III tell us that ϕ is the director circle of the inconic touching $l\alpha+m\beta+n\gamma=0$ at (α',β',γ')

This proves the first part of the general theorem of this paper, *viz* that the three circles are coaxal. Now, it remains to prove the second part, *viz.* the common radical axis envelopes the orthopolar conic of $(\alpha'\ \beta'\ \gamma')$.

Theorem II.—The radical axis of θ and ϕ is found to be

$$\alpha(p_1l^2+p_2m^2+p_3n^2)+\beta(q_1l^2+q_2m^2+q_3n^2)$$
$$+\gamma(r_1l^2+r_2m^2+r_3n^2)=0 \quad \ldots \quad \ldots \quad \ldots \quad \text{(M.)}$$

where

$$p_1=\frac{\alpha'}{\sin A}\Big\{\alpha' \sin B \sin C\ (\beta'+\gamma' \cos A)\ (B' \cos A+\gamma')$$
$$-\cos A\ \Sigma\beta'\gamma' \sin A.\quad \Sigma\alpha' \sin A\Big\}$$
$$p_2=\alpha'\beta' \sin C\ (\beta'+\gamma' \cos A)\ (\beta' \cos A+\gamma')$$
$$p_3=\alpha'\gamma' \sin B\ (\beta'+\gamma' \cos A)\ (\beta' \cos A+\gamma');$$

and symmetrical values for q and r.

We have now to find the envelope of M subject to the condition $l\alpha'+m\beta'+n\gamma'=0$. The envelope is easily seen to be of the form

$$BC\alpha' \sin A+CA\beta' \sin B+AB\gamma' \sin C=0, \quad \ldots \quad \ldots \text{(N)}$$

where

$$\frac{A\alpha'}{\sin A}\equiv\alpha p_1+\beta q_1+\gamma r_1$$
$$\frac{B\beta'}{\sin B}\equiv\alpha p_2+\beta q_2+\gamma r_2$$
$$\frac{C\gamma'}{\sin C}\equiv\alpha p_3+\beta q_3+\gamma r_3.$$

The relation (N) shews that the envelope which is evidently a conic is circumscribed to the triangle formed by the three straight lines A, B, C. And on finding the normal coordinates of the vertices of the triangle formed by A, B, C they are easily found to be the *former* set of points in Lemma IV.

Hence the whole theorem is proved.

Euclid's Book on Divisions of Figures. *

Euclid, it is now well known, was the author of no less than nine works. Four of these—the *Elements*, the *Data*, the *Optics*, and the *Phenomena*, have been carefully preserved and are well known, even to the extent that the first, perhaps, is in the hands of every schoolgirl. On the fifth, the *Porisms*, there is an elaborate commentary of Pappus, while in the case of the *Pseudaria*, the *Surface loci* and the *Conics* however, our fragmentary knowledge, derived wholly from Greek sources, enables us to make a conjecture of the vaguest nature, as to their possible contents. Of the ninth book, *On Division* (*of figures*), nothing however was known till about the middle of the last century ; nor, curiously enough, is there any trace of it in the extant Greek sources of mathematics, except perhaps a casual mention by Proclus. But it is however certain, that the Problems of this book, thirty-six in number, concerned mainly with the division of rectilinear figures and the circle into a certain number of 'like' and 'unlike' parts by straight lines, was very popular among Arabians, as well as in Europe. This popularity was perhaps largely due to the possible practical applications of the problems in the division of parcels of lands of various shapes, the areas of which, according to the Rhind papyrus, were discussed in empirical fashion about 1800 B. C. Indeed, in the first century before Christ, we find that Heron of Alexandria, dealt with the division of surfaces and solids in the third book of his *Metrika*, (*Surveying*) though not in the actual form in which Euclid dealt with the same. But nothing in a definite form was known until Woepcke, whose name is so familiar to every oriental researcher, discovered at the *Bibliotheque Nationale* in Paris, a treatise in Arabic on the division of plane figures, which he translated (into French) and published in the *Journal Asiatique*, September 1851, (XVIII), under the title "*Notice sur des traductions Arabes de deux ouvrages perdus d'Euclide*". In the original Arabic not only is the full account given (though unfortunately the proofs, save of four theorems, are passed over as being too easy !), but the scribe expressly attributes the authorship to Euclid, and contents himself with the unassuming office of a 'translator'. Such a treatise, significant alike for its historical interest as for its technical import, had to be restored and re-edited, if for no other reason, at least as a mark of our regard fro the author. Prof. Heiberg gives an account of the book only in a Latin publication, *Opera Omnia*, while Sir Thomas Heath, the foremost living authority in England on Greek Mathematics, dismisses the work in the brief space of two pages in his *Thirteen Books of Euclid's Elements* ; there is no reference to it even in the article on Euclid in the last

* By R. C. Archibald, Ph. D. Cambridge University Press, 1915.

edition of the *Encyclopœdia Brittanica.* Under these circumstances, Dr. R. H. Archibald of Providence, Rhode Island, has done a lasting service to the English reader in restoring in the English language this long buried treasure; and the scholarly way in which he has done his task is proved by the readiness with which the publication of the book was undertaken by the Cambridge University Press.

To follow in historical sequence how this treatise came to be unearthed or to dismiss the myth of Dee's making a copy of any Cottonian MSS earlier than 1563, is not our purpose in this article; the interested reader must be referred to the learned discussion in Archibald's book itself. But it must be noticed to what extent Woepcke's translation referred to above, is in agreement with and is supplemented by the writings of *Leonardo Pisano* (*Fibonaci*), who occupies such an important place in the history of Mathematics of the 13th century. It is well known that in 1220 he wrote his *Practica Geometriae*, a book largely used by the Mathematicians of the middle ages, published nearly 650 years after its compilation by Prince Boncompagni, and still preserved in MS in the Vatican library. Section IV of this book is wholly devoted to the enunciation and proof and numerical illustration of propositions concerning the division of figures. No less than twenty-two of Woepcke's propositions are practically identical in statement with propositions in Leonardo; the solutions of eight more of Woepcke are either given or are clearly indicated by Leonardo, and all the six of the remaining Woepcke's are assumed as known in the book of Leonardo. Indeed the two works have a remarkable similarity. Not only are practically all of Woepcke's theorems in Leonardo, but proofs called for by the order of the theorems and by the auxiliary propositions are invariably the kind of proofs which Euclid might have given—no other propositions than those which had gone before or which were to be found in the *Elements* being regarded in the successive constructions.

In this treatise of Euclid, the figures divided are the triangle, the parallelogram, the trapezium, the quadrilateral, the circle and a figure formed by the arc of a circle and two lines; and the division is by transversals through assigned points or parallel to given lines into two or several parts in a given ratio, equality included. It is difficult to choose from this interesting collection but the three following are specially instructive, and suggest generalisations. Thus Steiner solved in 1827 how *through a given point on a sphere to draw arc of a great circle cutting two given great circles such that the intercepted area is equal to*

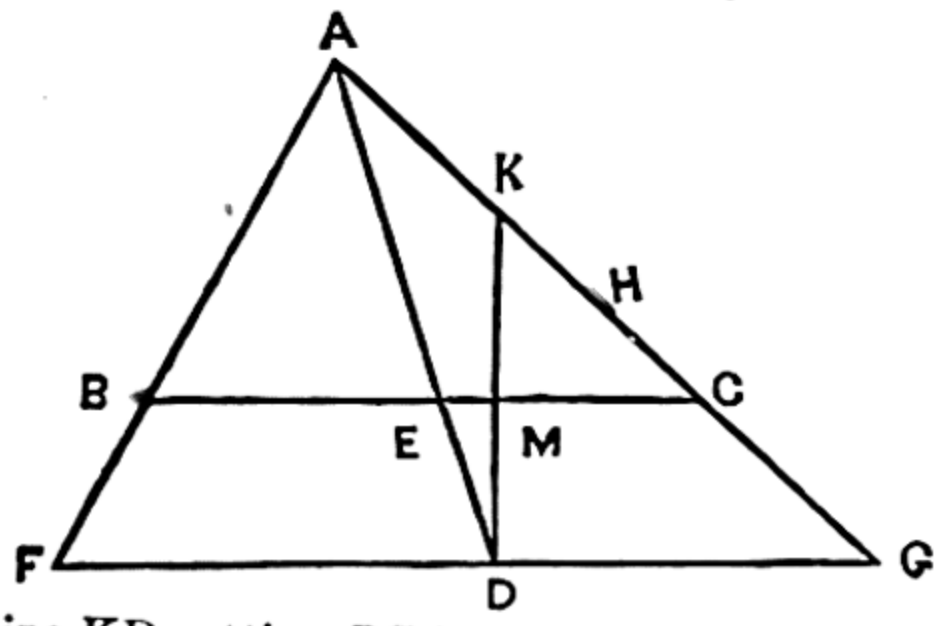
given area; and Bobillier solved: *to draw by means of planes and spheres only, a given line in a plane which shall cut off from a given cone of revolution a required volume.* The three theorems referred to above are given as propositions 27, 28 and 29 by Woepcke and are extracted below:

Proposition 27.

To cut off a certain fraction of a triangle by a straight line drawn from a given point situated outside of the triangle:

Let ABC be the given triangle and D, the given point outside. It is required to cut off from the triangle a certain fraction. say a third, by a line through D. Join AD cutting BC in E. If either BE or EC be a third of BC, then the line AD cuts off the required fraction from. the given triangle. If it is not, produce AB and AC to meet in F and G respectively a line drawn through D parallel to BC. Make area $DG \cdot CH = \frac{1}{3}$ area $AC \cdot CB$ and apply to the line CH, a rectangle equal to the rectangle $GC \cdot CH$, but exceeded by a square. Then

$$GC \cdot CH = HK \cdot KC$$



and draw the line KD cutting BC in M.

Then KMC shall be a third of the triangle ABC.

Proof. For since area $GC.CH =$ area $CK \cdot KH$,

and therefore $GC : CK = KH : HC$,

it follows that $GK : CK = CK : CH$,

But $GK : KC = DG : CM$,

Hence $DG \cdot CM = CK : CH$

i.e. $CK \cdot CM = CH \cdot DG$.

But area $DG \cdot CH = \frac{1}{3}$ area $AC \cdot CB$,

since $\triangle KCM : \triangle ACB = CK \cdot CM : CA \cdot CB$, it follows that $\triangle KCM = \frac{1}{3} \triangle ACB$.

In a similar manner any part of a triangle may be cut off by a straight line drawn from a given point on a side of the triangle produced, &c.

Q.E.F.

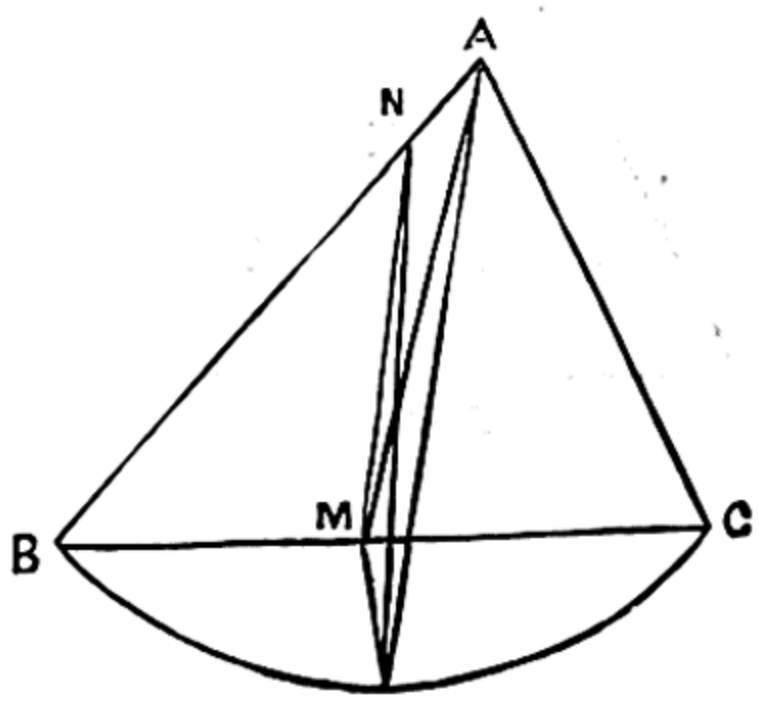
Proposition 28.

To divide into two equal parts a given figure bounded by an arc of a circle and by two straight lines which form a given angle.

Let ABC be the given figure bounded by the arc BC and two lines AB and AC which include the angle BAC. It is required to divide by means of a straight line the figure ABC into two equal parts.

Construction. Bisect the line BC at M, and erect MZ perpendicular to BC join MA. If AMZ are in a line, obviously the figure is divided into two equal parts by AZ. If not; join AZ, and through M draw a line parallel to AZ cutting AB at N; and join NZ.

Then NZ shall be the required line, and the figure BNZ shall be equal in area to the figure NACZ.



Proof. Obviously, $\Delta ABM = \Delta ACM$.

and figure BMZ = figure CMZ.

Hence figure ABZM = figure ACZM.

But $\Delta MAZ = \Delta NAZ$.

Therefore, figure ACZN = figure ACZ + ΔNAZ.

= figure ACZ + ΔMAZ.

= figure ACZM,

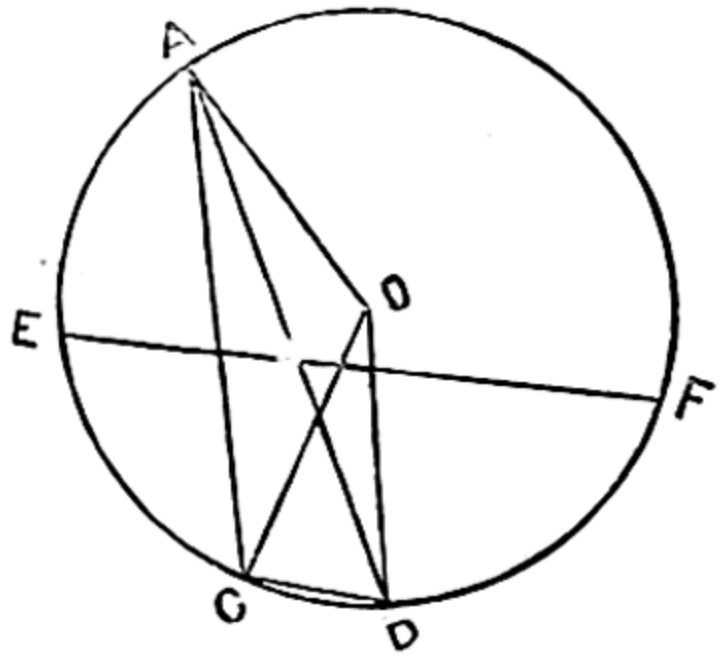
= $\frac{1}{2}$ the figure ABZC.

Q E, F.

Proposition 29.

"*To draw in a given circle two parallel lines cutting off a certain fraction, say, a third, from the circle.*

Take the circle with centre O, and inscribe in it the equilateral triangle ABC. Join AO and CO., and draw OD parallel to AC. Join CD. Bisect the arc AC at E and draw EF parallel to CD. Then shall the figure EC DF be a third of the given circle. Join AD.



Proof. Since AC and OD are parallel,

$$\triangle ACO = \triangle ACD ;$$

and, adding to each side the figure AEC, the figure AECD = sector ACO = $\frac{1}{3}$ of the circle.

Again, since EF and CD are parallel,

arc DF = arc EC

= the equal arc AE.

Add are ED to these.

arc AD = arc EF,

and chord AD = chord EF ;

and segment AED = segment ECDF

Take away the common segment CD,

and the figure ECDF = figure AECD = $\frac{1}{3}$ of the circle,

as was required to be shown.

Q. E. F.

In the Arabic Manuscript it seems to have been stated that the method is also available, when the fraction is *any* fraction. But Dr. Archibald hastens to add in a footnote that this problem is clearly not susceptible of solution with ruler and compasses, in such a case as when a

'certain fraction" $1/n$, is one-seventh. In fact the only cases in which the problem is possible, for a fraction of this kind, is when n is of the form $2^p(2^{2^{s_1}}+1)(2^{2^{s_2}}+1)\ldots(2^{2^{s_m}}+1)$ where p, and s's (all different), are positive integers or zero, and $2^{2^{s_m}}+1 (m=1, 2,\ldots m)$ is a prime number.

It is no wonder that such interesting theorems were considered independently or by suggestion by mathematicians, great and small. Thus the problems "*To bisect, by a straight line, an equilateral, but not equiangular heptagon*", "*Through a point without a triangle to draw a line which will cut off a third*" went as a challenge from L. F. (L. Ferari) to N. T (N. Tartaglia). Kepler, in his investigations of the path of a planet was led in 1609 to the problem "*To divide the area of a semicircle in a certain ratio by a straight line drawn through a given point on the diameter or on the diameter produced*" and this was attacked among others by Wallis, Cassini, Lagrange and Laplace. Newton and Bernoulli, D'Alembert and Euler, Huyghens and L'Hospital attacked several of the problems that have been given in this treatise of Euclid. These theorems, or their immediate generalisations have also fascinated the readers of popular magazines: thus in 1814, in the *Gentlemen's Diary* was solved "*within a given triangle to find thus, that if lines be drawn from it to cut each side at right angles, the three parts into which the triangle thus becomes divided, shall obtain a given ratio*"; while a century later in 1914 we find in the two most popular mathematical journals, the problem how "to *divide a circle into two equal parts by means of an arc with its centre on the circumference of the given circle*" (Q. 4327, *L'Intermediare des Mathematiciens*) and "*to divide a square into five right angled triangles the areas of which shall be in Arithmetic progression*" (Q. 17197 *Educational Times*). Among Indians who interested themselves in such problems, Dr. Archibald mentions the name of the late Mr. S. Radhakrishna Aiyar of Pudukotah.

SHORT NOTES.

On Legendre's relation in Elliptic Functions.

§ 1. The relation $EK'+E'K-KK'=\frac{\pi}{2}$,

connecting the two kinds of complete Elliptic integrals was first given by Legendre in his *Exercises de Calcul*, and several proofs of it are well known. [Cf. Dixon: *Elliptic Functions* § 65; Greenhill: *Elliptic Functions* § 121; Whittaker and Watson: *Modern Analysis*, § 22·735, Appellet Lacour: *Fonctions Elliptiques*, § 80; or Jacobi: *Fundamenta Nova*, p. 161]. Following however, the proof (cf. Forsyth: *Theory of Functions*, § 129 or Henry: *Fonctions Elliptiques*, p. 53) of the analogous relation in the Weierstrassian theory, viz, $\eta_1\omega_2-\eta_2\omega_1=\frac{\pi i}{2}$, we may proceed directly as follows.

We assume of course that

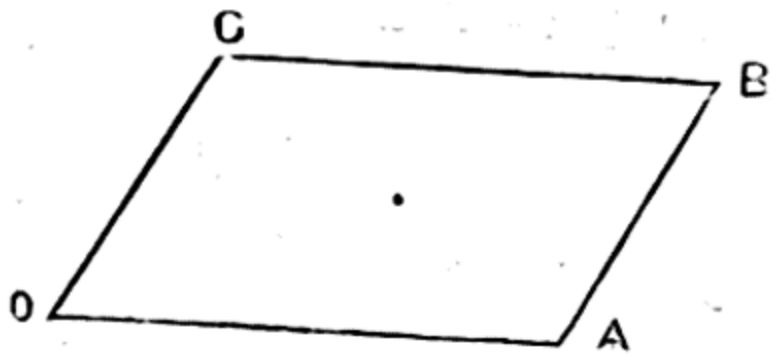
$$E(u)=\int_0^u \mathrm{dn}^2\, u\, du$$

and that the function dn u is a doubly periodic function of u with periods $2K$ and $4iK'$, that it is analytic except at the points congruent to iK' and to $3iK'$ which are simple poles, the residues thereat being $-i$ and $+i$ respectively.

It follows that $\mathrm{dn}\, u=-i/u+O(u)$ at $u=iK'$

$$\mathrm{dn}^2\, u=-1/u^2+O(1)$$

whence we notice that $E(u)$ has got a simple pole with residue 1 at $u=iK'$ which is the only pole in the parallelogram $[-K, K, K+2iK', -K+2iK']$



Integrating now $E(u)$ along the parallelogram OABC, we have

$$\int_{ABCO} E(u)du=2\pi i\times \text{the sum of the residues}=2\pi i,$$

$$i.e\int_{-K}^{K} E(u)du+i\int_{0}^{2K} E(K+iv)dv+\int_{K}^{-K} E(u+2iK')du \\ +i\int_{2K'}^{0} E(-K+iv)dv \Big\} =2\pi i,$$

$$i.e., \quad \int_{-K}^{-K}[E(u+2iK')-E(u)]du \\ -i\int_{0}^{2K'}[E(-K+iv)-E(K+iv)]dv \qquad =2\pi i$$

$$i.e., \quad \int_{-K}^{-K} 2i(K'-E')du+i\int_{0}^{2K'} 2\,E\,du \qquad =2\pi i$$

by the periodicity formulæ, cf. Dixon *Elliptic Functions* § 64.

$$i\,e, \qquad 4\,i\,(E\,K'+E'K-K\,K')=2\pi i$$

which establishes Legendre's relation connecting the two complete Elliptic Integrals.

§ 2. In passing, it may also be noticed that the above relation is only a special case of a more general result in the theory of Abelian functions (Cf. Baker: *Abelian Functions* p.p. 197—200): that between the modulii of periodicity A', B',...of a function P (z) of the first kind (Normal function on a Riemann Surface) and A, B,...of a function Q (z) of the second kind, all the infinities of which are simple,

$$\sum_{r=1}^{p}(A_r B_r'-B_r A_r')+2\pi i\Sigma H_s\left(\frac{dP}{dz}\right)_{a_s}$$

where a_s is a simple pole of $Q(z)$ and H_s its residue thereat. Legendre's relation is of course included in this: for, we know $P(z)\equiv F(z)$ in the notation of the Elliptic Functions

$$=\int_0^z[(1-z^2)(1-k^2z^2)]^{-\frac{1}{2}}dz$$

$$\therefore \frac{dP}{dz}=\frac{1}{K} \text{ in the lower sheet, and} -\frac{1}{K} \text{ in the upper for } Z=\infty$$

and $$Q(z)\equiv E(z)=\int_0^z\left[\frac{1-K^2Z^2}{1-Z^2}\right]^{\frac{1}{2}}dz.$$

This is infinite at $z=\infty$ in each sheet, and the residue is $+K$ in the upper and $-K$ in the lower sheet, so that $\Sigma H_s\left(\frac{dP}{dz}\right)_{a_s}=-2$.

Moreover A_1' and B_1' are $4K$ and $2iK'$

A_1 and B_1 are $4\,E$ and $2i(K'-E')$, respectively.

Hence substituting in the above we have immediately

$$4\,E.\,2iK'-4\,K.\,2i(K'-E')-4\pi i=0,$$

which leads to $$EK'+KE'-KK'=\pi/2.$$

K. B. Madhava.

Astronomical Notes.

Eclipse of the Sun.

The third and last eclipse of the year will occur on December 3. This will be an annular eclipse of the sun. The central line runs from a point just inside the west coast of Africa to a point in the Pacific Ocean, crossing the Argentine Republic. The eclipse is partially visible over nearly the whole of South America and a considerable portion of Africa. The southern limit does not quite reach the South Pole.

Nova Aquilae.

The new star in Aquila was discovered by about a dozen different observers in England within a few hours of one another, and also by numerous other observers in different parts of the world.

Both as regards magnitude and changes in its spectrum, it seems to have followed pretty close the changes observed in other novae. Perhaps the most interesting point about it is that it appears to be identical with a star which has been photographed several times during the last quarter of a century, the magnitude of which appears to have varied from the eighth to the tenth magnitude. With the exception of the nova known as T Coronæ, (which was also exceptional in that it did not appear in the Milky Way), there is no other case in which a star is known to have existed previously in the place where a nova has appeared,

New Comet.

The first discovery of a comet this year was made by Mr Reid at the Cape of Good Hope in June, when it appeared as a faint round nebulosity. As yet only very meagre details are to hand, so that it is impossible to give any further information at present.

New Trojan Planet.

A new planet having the same mean distance as Jupiter and thus illustrating Lagranges particular Solution of problem of the three bodies has been discovered. This is the fifth such planet known, three of the planets librate in the neighbourhood of the triangular point on one side of the line joining the Sun and Jupiter, and the other two, of which the new discovery is one, librate about the apex of the equilateral triangle on the opposite side of the Sun—Jupiter line.

R. J. Pocock.

SOLUTIONS.

Question 685.

(S. Malhari Rao) :—If n and p are any two positive integers, show that the fraction $\frac{1}{n}$ may be expressed as the sum of a series of p different fractions, the numerator of each of which is 1 and the denominator a positive integer. If the least of the fractions be $(n-1)!/(n+p-1)!$ find the rth fraction.

Solution by Prof. N. B. Mitra and N. Sankara Aiyar M. A.

The question is vague; for, $\frac{1}{n}$ can be resolved into the sum of p unit-numerator fractions in several ways. Thus, one way would be as follows :—

We easily see that $$\frac{1}{n}=\frac{1}{n(n+1)}+\frac{1}{n+1}. \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

By applying the same formula to $\frac{1}{n+1}$, the second fraction on the right, we have,

$$\frac{1}{n}=\frac{1}{n\,(n+1)}+\frac{1}{(n+1)\,(n+2)}+\frac{1}{n+2}.$$

By repeating the same process we get,

$$\frac{1}{n}=\frac{1}{n\,(n+1)}+\frac{1}{(n+1)\,(n+2)}+\ldots\ldots+\frac{1}{(n+p-2)\,(n+p-1)}+\frac{1}{n+p-1}.$$

Another way is to substitute in the first fraction on the right of (1), the value of $1/n$ given by (1). If this is repeated $p-1$ times we get ultimately

$$\frac{1}{n}=\frac{1}{n\,(n+1)^{p-1}}+\frac{1}{(n+1)^{p-1}}+\frac{1}{(n+1)^{p-2}}+\ldots\ldots+\frac{1}{n+1}.$$

A third way is as follows:—In (1) apply the same formula to $\frac{1}{n+1}$ in the first fraction on the right. We get

$$\frac{1}{n}=\frac{1}{n\,(n+1)\,(n+2)}+\frac{1}{n\,(n+2)}+\frac{1}{n+1}.$$

By applying the process altogether $p-1$ times, we get

$$\frac{1}{n}=\frac{1}{n\,(n+1)\ldots(n+p-1)}+\frac{1}{n\,(n+1)\ldots(n+p-3)\;(n+p-1)}$$
$$+\frac{1}{n\,(n+1)\ldots(n+p-4)\;n+p-2)}+\ldots+\frac{1}{n\,(n+2)}+\frac{1}{n+1}$$
$$=\lfloor n-1\left[\frac{1}{\lfloor n+p-1}+\frac{1}{\lfloor n+p-3\;(n+p-1)}\right.\right.$$
$$\left.\left.+\frac{1}{\lfloor n+p-4\;(n+p-2)}+\ldots+\frac{1}{\lfloor n-1\;(n+1)}\right]\right.$$

The r^{th} fraction on the right is evidently $\frac{\lfloor n-1}{\lfloor n+p-r-1\;(n+p-r+1)}$

the least being the first.

This is perhaps what the proposer seeks by the second part of his question.

Question 954.

(K. B. Madhava):—When asked his age De Morgan once humorously observed: "I am one of those whose age shall be in a certain year belonging to the century of their birth the square root of that year. "For being born in 1806 A. D., he was 43 years old in 1849 and 1849=43. Show that the same observation will be true in the case of those born in 9 particular years in all subsequent centuries.

Solution by N. B. Mitra, K. J. Sanjana and R. D. Karve.

It would appear that the figure 9 in the last line of the question is wrong. The problem may be solved thus:—Let x be the year of birth, y the age at which the year is square of the age. Then we have to solve the indeterminate equation $y^2=x+y$ in non-zero positive integers subject to the conditions (1) that $x>1900$ and (2) that $y+$ Least positive residue of x, modulus $100>100$ this residue itself being >0. This gives 15 sets of values of x and y which are exhibited below:—

$x=2450$; 3306 ; 3422 ; 3540 ; 3906 ; 4032 ; 4422 ; 4830 ; 5112 ;
5402 ; 6006 ; 6320; 6806 ; 7310 ; 8010.

$y=50$; 58 ; 59 ; 60 ; 63 ; 64 ; 67 ; 70 : 72 ; 74; 78 ; 80 ; 83 . 86 ; 90.

It may be interesting to know the years in the centuries previous to the 20th in which De Morgan's observation was true. They are given below together with the corresponding ages at which the year of the Christian era was square of the age :—

$x=2$; 6 ; 12 ; 20 ; 30 ; 42 ; 56 ; 72; 90 ; 110 ; 132 ; 156 ;
182 ; 210 ; 240 ; 272 ; 306 ; 342 ; 380 ; 420 ; 462 ; 506 ; 552 ;

$y=2$; 3 ; 4 ; 5 ; 6 ; 7 ; 8 ; 9 ; 10 ; 11 ; 12 ; 13 ; 14 ; 15 ; 16 ;
17 ; 18 ; 19 ; 20 ; 21 ; 22 ; 23 ; 24 ;

$x=690$; 702 ; 756 ; 812 ; 870 ; 930 ; 1056 ; 1122 ; 12600 ;
1332 ; 1406; 1560 ; 1640 ; 1722 ; 1806 (De Morgan's case.)

$y=26$; 27 ; 28 ; 29 ; 30 ; 31; 33 ; 34 ; 36 ; 37 ; 38 ; 49 ; 41 ; 42 ; 43 ;

Question 831.

(Enquirer) Find, by elementary methods, the invariant relation expressing the condition that hexagons may be inscribed in the conic $S=0$ which are also circumscribed to the conic $S'=0$.

If a hexagon ABCDEF circumscribes a conic S and is inscribed in S', then the Brianchon point of the Pascal line of the hexagon are pole and polar.

Solution by A. C. L. Wilkinson.

Projecting S' into a circle with the Brianchon point as centre we have hexagons inscribed in $S' \equiv x^2+y^2-c^2=0$ and circumscribed to

$$S \equiv \frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0.$$

To find the relation between a, b, c consider that hexagon ABCDEF of which A and D lie on the minor axis of the ellipse and BC, EF touch the ellipse at the extremities of the major axis.

Let AB touch the ellipse at $(a\cos\alpha, b\sin\beta)$ then A is $\left(0, \frac{b}{\sin\alpha}\right)$ and B is $\left(a, b\tan\frac{\alpha}{2}\right)$.

Hence $c^2=a^2+b^2\tan\frac{2a}{2}=b^2\operatorname{cosec}^2\alpha$

giving $(c^2-a^2+b^2)^2=4c^2(c^2-a^2)$

or $3c^4-2c^2(a^2+b^2)-(a^2+b^2)^2+4a^2b^2=0$ ... ... (1)

Also $\Delta=-\frac{1}{a^2b^2}$, $\Theta=-\frac{1}{a^2b^2}(a^2+b^2+c^2)$, $\Theta'=-\frac{a^2b^2+a^2c^2+b^2c^2}{a^2b^2}$, $\Delta'=-c^2$

write $p=\frac{\Theta}{\Delta}=a^2+b^2+c^2$, $q=\frac{\Theta}{\Delta}a^2b^2+a^2c^2+b^2c^2$, $v=\frac{\Delta'}{\Delta}=a^2b^2c^2$.

Then a^2, b^2, c^2 are the roots of

$$x^3-px^2+qx-r=0$$

and from (1) $$3x^2-2x(p-x)-(p-x)^2+\frac{4r}{x}=0$$

or $$4x^3-p^2x+4r=0.$$

Write $2p$ for p, so that $x^3-2px^2+qx-r=0$, $x^3-p^2x+r=0$;

whence

$$2x^2-2px+q-p^2=0 \text{ and } 2px^2-(p^2+q)x+2r=0;$$

which gives

$$\{(p^4-q^2)+4pr\}(p^2-q)+2\{p^2-pq+2r\}^2=0,$$

or $$(3p^2+q)(p^2-q)^2+12pr(p^2-q)+8r^2=0,$$

Hence the invariant relation is

$$(3\theta^2+4\Delta\theta')(\theta^2-4\Delta\theta')^2+96\theta\Delta^2\Delta'(\theta^2-4\Delta\theta')+512\Delta'^3\Delta^4=0$$

This will be found to agree with $\begin{vmatrix} D, & E \\ E, & F \end{vmatrix}=0$ of Salmon after removing the factor $\theta^2-4\Delta\theta'$, and is easily identified with Halphen's form which is

$$4(\theta^3-4\Delta\theta\theta'+8\Delta^2\Delta')^2+4(\theta^3-4\Delta\theta\theta'+8\Delta^2\Delta').8\Delta^2\Delta'-(\theta^2-4\Delta\theta')^3=0.$$

Question 844.

(A. A. KRISHNASWAMIENGAR, M. A.) :—Shew that the locus of the centre of the conic of minimum eccentricity having contact of the third order with the conic $Ax^2+By^2=1$, has for its equation $(x^2+y^2)\sqrt{(Ax^2+By^2)}=(Ax^2-By^2)\left(\frac{1}{A}-\frac{1}{B}\right)$

Solution (1) *by N. B. Mitra, Hemraj, and S. V. Venkatachala Aiyar ;*
(2) *by F. H. V. Gulasekharam.*

(1) The equation giving the eccentricity of the general conic in rectangular coordinates is $\frac{e^4}{1-e^2}=\frac{(a+b)^2}{ab-n^2}-4$

If $e<1$, e will be least when $\frac{(a+b)^2}{ab-n^2}$ is least since $\frac{e^4}{1-e^2}$ in this case diminishes continually with e.

Now the conic having four pointic contact with $Ax^2+By^2=1$...(1) at the point $\left(\frac{\cos\theta}{\sqrt{A}}, \frac{\sin\theta}{\sqrt{B}}\right)$ has for its equation

$$Ax^2+By^2-1-\lambda(\sqrt{A}x\cos\theta+\sqrt{B}y\sin\theta-1)^2=0 \quad \ldots \quad \ldots \quad (2)$$

The eccentricity of this is least when

$$\frac{\{A(1-\lambda\cos^2\theta)+B(1-\lambda\sin^2\theta)\}^2}{AB(1-\lambda\cos^2\theta)(1-\lambda\sin^2\theta)-AB\sin^2\theta\cos^2\theta}$$

or $\frac{\{A+B-\lambda(A\cos^2\theta+B\sin^2\theta)\}^2}{AB(1-\lambda)}$ or $\frac{(A\cos^2\theta+B\sin^2\theta)^2}{AB(1-\lambda)}$

$$\left\{\frac{A\sin^2\theta+B\cos^2\theta}{A\cos^2\theta+B\sin^2\theta}+1-\lambda\right\}$$

is least or putting $p=A\sin^2\theta+B\cos^2\theta$ & $q=A\cos^2\theta+B\sin^2\theta$, when

$$\frac{q^2}{AB}\left\{\frac{\frac{p^2}{q^2}}{(1-\lambda)}+2\frac{p}{q}+(1-\lambda)\right\} \text{ is least } i.e. \text{ when } 1-\lambda=\frac{p}{q}$$

$$\text{or}\lambda=\frac{(A-B)\cos 2\theta}{q} \quad \ldots \quad (3).$$

The centre of (2) is given by $A(1-\lambda\cos^2\theta)x-\lambda\sqrt{AB}\cos\theta\sin\theta y+\lambda\sqrt{A}\cos\theta=0.$

$$\&-\lambda\sqrt{AB}\cos\theta\sin\theta x+B(1-\lambda\sin^2\theta)y+\lambda\sqrt{B}\sin\theta=0.$$

i.e. by $\lambda\cos\theta(\sqrt{A}x\cos\theta+\sqrt{B}y\sin\theta-1)=\sqrt{A}.x$...(4).

and $\lambda\sin\theta(\sqrt{A}x\cos A+\sqrt{B}y\sin\theta-1)=\sqrt{B}.y$...(5).

Multiplying (4) by $\cos\theta$ & (5) by $\sin\theta$ and adding

$$\lambda(\sqrt{A}x\cos\theta+\sqrt{B}y\sin\theta-1)=\sqrt{A}x\cos\theta+\sqrt{B}y\sin\theta$$

$$\text{or } \sqrt{A}x\cos\theta+\sqrt{B}y\sin\theta=\frac{\lambda}{\lambda-1}=\frac{(B-A)\cos^2\theta}{p}\text{ from (3)} \quad \ldots \quad (6)$$

also from (4) and (5) $\dfrac{\sqrt{A}x}{\cos\theta}=\dfrac{\sqrt{B}y}{\sin\theta}=\sqrt{\dfrac{Ax^2+By^2}{1}}=K$ suppose.

$$\therefore\ \cos\theta=\frac{\sqrt{Ax}}{k},\ \sin\theta=\frac{\sqrt{By}}{k}\ \therefore\ \cos^2\theta=\frac{Ax^2-By^2}{k^2}$$

$$\text{and } p=\frac{AB(x^2+y^2)}{k^2}$$

$$\therefore\ \text{from (6)},\ \frac{Ax^2+By^2}{k}=\frac{(B-A)(Ax^2-By^2)}{AB\ (x^2+y^2)}\ \text{or}\ (x^2+y^2)\sqrt{(Ax^2+By^2)}$$

$$=\left(\frac{1}{A}-\frac{1}{B}\right)(Ax^2-By^2)$$

[See: Wolstenholme, Ex. 1209, p. 202.]

(2) The equation to the conic having contact of the third order with $Ax^2+By^2=1$ at any point P (x_1, y_1) on it is

$$Ax^2+By^2-1-k(Axx_1+Byy_1-1)^2=0 \quad \ldots \quad \ldots \quad (1)$$

The coordinates (x, y) of the centre Q of this conic are given by the equations,

$$\frac{x}{x_1}=\frac{y}{y_1}=k(Axx_1+Byy_1-1)$$

$$=\lambda \text{ say} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

Then since (x_1, y_1) is a point on the given conic,

$$\left.\begin{aligned}\lambda^2&=Ax^2+By^2\\ k&=\frac{\lambda}{\lambda-1}\end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

and

Also from (2), it is clear that the centre of the conic (1) lies on the diameter through P of the given conic.

Again from Ex. 6 Chap XI, C. Smith's Conic Sections (Revised Edition), "Of all conics which have a third order contact with a given conic at a given point on it, the one of least eccentricity will be that in which one of the equi-conjugate diameters passes through that point."

Hence if r_1, r_2 be the semi—axes of the conic (1),

$$2PQ^2=r_1^2+r_2^2;$$

$$\therefore 2[(x-x_1)^2+(y-y_1)^2]=\frac{A+B-k(A^2x_1^2+B^2y_1^2)}{AB(1-k)}$$

(Use the equation to determine the length of the axes of a central conic given by the general equation of the second degree.)

Now substituting for x_1, y_1 and k from (2) and (3), the above equation becomes.

$$2\left[1-\frac{1}{\lambda}\right]^2(x^2+y^2)=\frac{A+B-\frac{A^2x^2+B^2y^2}{\lambda(\lambda-1)}}{\frac{AB}{(\lambda-1)^2}},$$

which, when simplified and Ax^2+By^2 is written for λ^2, reduces to

$$(x^2+y^2)\sqrt{Ax^2+By^2}=\left(\frac{1}{A}-\frac{1}{B}\right)(Ax^2-By^2)$$

giving the equation of the locus required.

Question 845.

(M. K. Kewalramani) :—ABC is a triangle such that cos 2 B, cos 2 A, cos 2 C are in A. P. A rectangular hyperbola circumscribes this triangle and passes through its centroid. AD is the common chord of this hyperbola and the osculating circle at A meeting the opposite side in D. show that AD divides BC in the ratio $\sin\overline{A-B}\,[b\sin\overline{B-C}-a\sin\overline{C-A}]$: $\sin\overline{C-A}\,[c\sin\overline{B-C}-a\sin\overline{A-B}]$.

Solution by Hemraj, S. V. Venkatachala Aiyar.

If $\cos 2B+\cos 2C=2\cos 2A$, it is easy to show that

$$b^2+c^2=2a^2,\ \frac{\sin(B-C)}{\sin(C-A)}=-2\frac{b}{a},\ \frac{\sin(B-C)}{\sin(A-B)}=-2\frac{c}{a}\quad\ldots\qquad\ldots\quad(1)$$

From the last two relations of (1), it follows at once that the ratio

$\sin\overline{A-B}\,[b\sin\overline{B-C}-a\sin\overline{C-A}]$:

$\sin\overline{C-A}\,[C\sin\overline{B-C}-a\sin\overline{A-B}]=2b^2+a^2:\ 2c^2+a^2$.

So we have to show that AD divides CB in the ratio

$$2b^2+a^2:\ 2c^2+a^2.$$

Now take the triangle ABC as the triangle of reference, and the equation of the rectangular hyperbola is

$$uyz+vzx+wxy=0$$

with the conditions $bc \cos A\, u + ca \cos B\, v + ab \cos C\, w = 0$
and $u + v + w = 0$,

from which we have $\dfrac{u}{b^2-c^2} = \dfrac{v}{c^2-a^2} = \dfrac{w}{a^2-b^2}$.

Hence the equation of the rectangular hyperbola takes the form

$$(b^2-c^2)\, yz + (c^2-a^2)\, zx + (a^2-b^2)\, xy = 0.$$

A is (1, 0, 0) and the tangent thereat is $(a^2-b^2)\, y + (c^2-a^2)\, z = 0$; *i.e.* $y+z=0$ from the first relation of (1).

And any line through A is $\mu y + \nu z = 0$.

Hence the equation of a conic having a contact of the 2nd order at A with the hyperbola is

$$2(b^2-c^2)\, yz + 2(c^2-a^2)\, zx + 2(a^2-b^2)\, xy + (y+z)(\mu y + \nu z) = 0$$

It represents a circle if

$$\frac{\mu + \nu - 2\left(b^2-c^2 + \frac{1}{2}\mu + \frac{1}{2}\nu\right)}{a^2} = \frac{\nu - 2(c^2-a^2)}{b^2} = \frac{\mu - 2(a^2-b^2)}{c^2},$$

which gives

$$\mu = 2(a^2-b^2) - 2\frac{c^2}{a^2}(b^2-c^2) = \frac{2}{a^2}[(c^2-a^2)\, a^2 - c^2(b^2-c^2)]$$

$$= \frac{2}{a^2}(c^2-a^2)(2c^2+a^2)$$

and $\nu = 2(c^2-a^2) - 2\dfrac{b^2}{a^2}(b^2-c^2) = \dfrac{2}{a^2}[(a^2-b^2)\, a^2 - b^2(b^2-c^2)]$

$$= \frac{2}{a^2}(a^2-b^2)(2b^2+a^2) \text{ from (1).}$$

$$\therefore \quad \nu/\mu = (2b^2+a^2)/(2c^2+a^2).$$

Now the equation of the chord of curvature is evidently $\mu y + \nu z = 0$ and it divides CB in the ratio $\dfrac{-y}{z}$ *i.e.* $\dfrac{\nu}{\mu}$ *i.e.* $(2b^2+a^2)/(2c^2+a^2)$. Hence the result.

Question 862.

(Communicated by Mr. Hem Raj) :—Angles A, B, C of a triangle are trisected by AQ, AR ; BR, BP ; CP, CQ. Shew that the triangle PQR is equilateral.

Solution (1) *by F. H. V. Gulaskaram, H. R. Kapadia, Sodanand, K. R. Rama Iyer,* (2) *by Hemraj and S. V. Venkatachala Aiyar.*

(1) Let P, Q, R be such that $\hat{CBP} = \dfrac{B}{3}$, $\hat{PCB} = \dfrac{C}{3}$, $\hat{QOA} = \dfrac{C}{3}$,

$\hat{CAQ} = \dfrac{A}{3}$, $\hat{RAB} = \dfrac{A}{3}$, $RBA = \dfrac{B}{3}$.

Let BR, CQ meet at D ; CP, AR meet at E, and AQ, BP meet at F.
Hence P is the incentre of the $\triangle$ DBC

$$\therefore \quad BP = a \sin\frac{C}{3} \operatorname{cosec}\left(\frac{B+C}{3}\right)$$

$$= a \sin\frac{C}{3} \operatorname{cosec}\left(60 - \frac{A}{3}\right)$$

Also $$a = 2R \sin A = \frac{R}{2} \sin A/3 \sin\left(60 + \frac{A}{3}\right) \sin\left(60 - \frac{A}{3}\right)$$

$$\therefore \quad BP = \frac{R}{2} \sin\frac{A}{3} \sin\frac{C}{3} \sin\,(60 + A/3).$$

Similarly $$BR = \frac{R}{2} \sin\frac{A}{3} \sin\frac{C}{3} \sin\,(60 + C/3)$$

$$\therefore \quad PR^2 = \frac{R^2}{4}\left\{\sin^2\left(\frac{A}{3}\right)\sin^2\left(\frac{C}{3}\right)\right\}\left[\sin^2\left(60+\frac{A}{3}\right)\right.$$

$$\left. + \sin^2\left(60+\frac{C}{3}\right) - 2\cos\frac{B}{3}\sin\left(60+\frac{A}{3}\right)\sin\left(60+\frac{C}{3}\right)\right]$$

$$= \frac{R^2}{4} \sin^2\,(A/3) \sin^2\,(B/3) \sin^2\,(C/3).$$

$$\therefore \quad PR = \frac{1}{2} R \sin\frac{A}{3} \sin\frac{B}{3} \sin\frac{C}{3}.$$

Similarly $RQ = QP = \frac{1}{2} R \sin A/3 \sin B/3 \sin C/3$.

Hence the triangle PQR is equilateral.

We can similarly shew that the $\triangle$ DEF is also equilateral.

(2) Put $A = 3\lambda, B = 3\mu, C = 3\nu$. In ΔAQR,

$$\frac{\sin\alpha}{\sin\beta} = \frac{AR}{AQ} = \frac{c \sin\mu}{\sin(\lambda+\mu)} \times \frac{\sin(\lambda+\nu)}{b \sin\nu}$$

$$= \frac{\sin 3\nu \sin\mu \sin(\lambda+\nu)}{\sin 3\mu \sin\nu \sin(\lambda+\mu)}$$

$$= \frac{(3-4\sin^2\nu)\sin(60^\circ-\mu)}{(3-4\sin^2\mu)\sin(60^\circ-\nu)} = \frac{\sin\,(60^\circ+\nu)}{\sin\,(60^\circ+\mu)}$$

There arise three cases.

(*a*) $\alpha = 60^\circ + \nu$, $\beta = 60^\circ + \mu$.

(*b*) $\alpha = 60^\circ + \nu$, $\beta = 120^\circ - \mu$; or $\alpha = 120^\circ - \nu$, $\beta = 60 + \mu$; (inadmissible).

(*c*) $\sin\alpha = k_1 \sin\,(60^\circ + \nu)$, $\sin\beta = k_1 \sin\,(60^\circ + p_1)$.

It is easy to show that (*c*) is also inadmissible, as

$$\frac{\sin\alpha}{\sin\,(60^\circ+\nu)} = \frac{\sin\beta}{\sin\,(60^\circ+\mu)}$$

and $$\frac{AR}{\sin\alpha} = \frac{AQ}{\sin\beta} = \frac{QR}{\sin\lambda}, \quad \therefore \quad \frac{AR}{k_1 \sin(60^\circ+\nu)} = \frac{AQ}{k_1 \sin\,(60^\circ+p_1)} = \frac{QR}{\sin\lambda}$$

i.e., $$\frac{AR}{\sin\alpha'} = \frac{AQ}{\sin\beta'} = \frac{QR}{\sin\lambda}$$

Now $\alpha+\beta+\lambda=180°$ and $60°+\nu+60+\mu+\lambda=180°$, it follows that $\alpha'+\beta'+\lambda\neq 180°$ unless $k_1=1$

Hence (*a*) is the only admissible solution.

$$\therefore \quad \alpha=60°+\nu,\ \beta=60°+\mu,\ r=60°+\lambda$$

$$\angle RPQ=360°-\angle BPC-\alpha-\beta=60°=\angle PRQ=\angle PQR$$

$\therefore$ PQR is equilateral.

Question 913.

(Prof. K. J. Sanjana) :—If the expansion of $(\sec x+\tan x)$ is written in the form

$$1+S_1\frac{x}{1!}+S_2\frac{x^2}{2!}+S_3\frac{x^3}{3!}+\ldots\ldots$$

prove that

$$S_n=2^{n+2}\,I/\pi^{n+1}$$

where $I\equiv\int_0^a \frac{(\log\sec x)^n}{\sin x}\,dx$ or $\log(\sqrt{2}+1)\int_0^b(\log\operatorname{cosech} x)^n/\cosh x\,dx$,

according as n is odd or even and $a=\frac{1}{2}\pi$, $b=\log(\sqrt{2}+1)$.

Solution by Hemraj, R. Srinivasan, S. R. Ranganathan, C. Krishnamachary, H. R. Kapadia, A. K. Krishnaswami Iyer, and M. K. Kewalramani.

From Chap. XVII. Hobson's *Trignometry*, or Arts. 573, 574: *Diff. Cal.* by Edwards, we have

$$\sec x+\tan x=1+S_1\frac{x}{1!}+S_2\frac{x^2}{2!}+S_3\frac{x^3}{3!}+\ldots$$

where

$$S_n=\frac{2^{n+2}n!}{\pi^{n+1}}\left\{1+\left(-\frac{1}{3}\right)^{n+1}+\left(\frac{1}{5}\right)^{n+1}+\left(\frac{1}{7}\right)^{n+1}+\left(\frac{1}{9}\right)^{n+1}+\ldots\right\}$$

Hence we have to show that

$$(i)\int_0^a\frac{(\log\sec x)^{2n-1}}{\sin x}dx=(2n-1)!\left\{1+\frac{1}{3^{2n}}+\frac{1}{5^{2n}}+\ldots\right\}$$

$$(ii)\int_0^b\frac{(\log\operatorname{cosech} x)^{2n}}{\cosh x}dx=2n!\left\{1-\frac{1}{3^{2n+1}}+\frac{1}{5^{2n+1}}-\frac{1}{7^{2n+1}}+\ldots\right\}.$$

Question 919.

(C. KRISHNAMACHARY) :—Prove that

$$1-\frac{1}{3}-\frac{1}{5}+\frac{1}{7}+\frac{1}{9}-\frac{1}{11}-\frac{1}{13}+\ldots=\frac{1}{\sqrt{2}}\log(1+\sqrt{2})$$

Solution (1) *by K. J. Sanjana and H. R. Kapadia*; (2) *by S. R. Ranganathan, R. Srinivasan and G A. Kamtekar*, (3) *Hemraj, Sadanand and N. B. Mitra.*

Let $$S=x-\frac{x^3}{3}-\frac{x^5}{5}+\frac{x^7}{7}+\frac{x^9}{9}-\ldots;$$

then $\frac{dS}{dx}$

is the sum of two G.P.'s. Finding their sums and integrating we shall be able to evaluate S. The following is a trigonometrical method.

In Gregory's series change x to $x\sqrt{\iota}$ and multiply by $\sqrt{\iota}$; we get

$$\sqrt{\iota}\tan^{-1}(x\sqrt{\iota})=\iota\left(x-\frac{x^5}{5}+\frac{x^9}{9}-\ldots\right)+\left(\frac{x^3}{3}-\frac{x^7}{7}+\frac{x^{11}}{11}=\ldots\right)$$
$$=\iota S_1+S_2, \text{ suppose.}$$

Then $S=S_1-S_2$, and we have thus to separate $\sqrt{\iota}\tan^{-1}x\sqrt{\iota}$ into real and imaginary parts. Taking $+\sqrt{\iota}\equiv(1+\iota)/\sqrt{2}$, we have

$$\frac{1+i}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}+\frac{x_2}{\sqrt{2}}\right.$$
$$=\frac{1+i}{\sqrt{2}}\left\{\frac{1}{2}\tan^{-1}\frac{x\sqrt{2}}{1-\frac{1}{2}x^2-\frac{1}{2}x^2}+\frac{1}{2}i\tanh^{-1}\frac{x\sqrt{2}}{1+\frac{1}{2}x^2+\frac{1}{2}x}\right\}$$
$$=\frac{1+\iota}{2\sqrt{2}}\left\{\tan^{-1}\frac{x\sqrt{2}}{1-x^2}+i\tanh^{-1}\frac{x\sqrt{2}}{1+x^2}\right\},$$

by the usual procedure. This gives at once

$$S_1=\frac{1}{2\sqrt{2}}\left(\tan^{-1}\frac{x\sqrt{2}}{1-x^2}+\tanh^{-1}\frac{x\sqrt{2}}{1+x^2}\right),\ S^2=\frac{1}{2\sqrt{2}}$$
$$\left(\tan^{-1}\frac{x\sqrt{2}}{1+x^2}-\tanh^{-1}\frac{x\sqrt{2}}{1+x^2}\right);$$

whence

$$S=\frac{1}{2\sqrt{2}}\left\{2\tanh^{-1}\frac{x\sqrt{2}}{1+x^2}\right\}=\frac{1}{2\sqrt{2}}\log\frac{1+x\sqrt{2}+x^2}{1-x\sqrt{2}+x^2}.$$

The result in the question follows by making $x=1$, which is allowable as the signs are $-$ and $+$ in pairs.

We also get

$$S_1+S_2=\frac{1}{\sqrt{2}}\tan^{-1}\frac{x\sqrt{2}}{1-x^2},$$

so that

$$1+\frac{1}{3}-\frac{1}{5}-\frac{1}{7}+\frac{1}{9}+\frac{1}{11}-\ldots=\frac{\pi}{2\sqrt{2}}.$$

(2)

We have for $-\frac{\pi}{2}<x<\frac{\pi}{2}$

(*Bromwich*: Art. 90 Ex. 2).

$$e^{xi}-\frac{e^{3xi}}{3}+\frac{e^{5xi}}{5}\ldots=\frac{\pi}{4}+\tfrac{1}{2}i\log(\sec x+\tan x)$$

Equating the imaginary parts, we have,

$$\sin x-\frac{\sin 3x}{3}+\frac{\sin 5x}{5}-\ldots=\tfrac{1}{2}\log(\sec x+\tan x)$$

Putting $x=\frac{\pi}{4}$,

we get,

$$1-\frac{1}{3}-\frac{1}{5}+\frac{1}{7}+\frac{1}{9}-=-\sqrt{2}\log(\sqrt{2}+1)$$

$$=\frac{1}{\sqrt{2}}\log(\sqrt{2}+1)$$

(3) It follows from $\tan^{-1}(\cos\theta+i\sin\theta)$ *or* $\frac{1}{2}\log\left(\frac{1+Z}{1-Z}\right)$

that if $0<x<\frac{\pi}{2}$ $\sin x-\frac{1}{3}\sin 3x+\frac{1}{5}\sin 5x-\frac{1}{7}\sin 7x+\ \ldots\ \ldots$

$$=\tfrac{1}{2}\log(\sec x+\tan x)$$

Putting $x=\frac{\pi}{4}$ we have the result.

Questions 919, & 968.

919. (C. Krishnamachary):—Prove that

$$1-\frac{1}{3}-\frac{1}{5}+\frac{1}{7}+\frac{1}{9}-\frac{1}{11}-\frac{1}{13}+=\frac{1}{\sqrt{2}}\log(1+\sqrt{2})$$

968. (V. Anantaraman):—Sum the series

$$\frac{1}{3}-\frac{1}{7}+\frac{1}{11}-\frac{1}{15}+\frac{1}{19}\ldots$$

Solution (i) *by T. H. V. Gulasekharam, N. G. Leather and S. Muthukrishnan;* (ii) *L. S. Vaidyanathan and M. K. Kewalramani.*

(i) We have on page 303, Hobson's *Plane Trigonometry*, the formula

$$\log(2\cos\tfrac{1}{2}\theta)=\cos\theta-\tfrac{1}{2}\cos 2\theta+\tfrac{1}{3}\cos 3\theta-\tfrac{1}{4}\cos 4\theta+\ldots;$$

write $\theta=\frac{\pi}{4}$.

Then remembering that

$$\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots\ldots$$

we deduce

$$1 - \frac{1}{3} - \frac{1}{5} + \frac{1}{7} + \frac{1}{9} - \frac{1}{11} - \frac{1}{13} + \ldots = \frac{1}{\sqrt{2}} \log (1+\sqrt{2}) \ldots \qquad \ldots (1)$$

Again, we have the formula

$$\tfrac{1}{2}\theta = \sin\theta - \tfrac{1}{2}\sin 2\theta + \tfrac{1}{3}\sin 3\theta - \tfrac{1}{4}\sin 4\theta + \ldots\ldots ;$$

write $\theta = \frac{\pi}{4}$.

Then remembering

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\ldots$$

we deduce that

$$1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \frac{1}{13} - \ldots = \frac{\pi}{2\sqrt{2}} \qquad \ldots \qquad \ldots \quad (2)$$

Hence subtracting (1) from (2) and dividing the result by 2,

$$\frac{1}{3} - \frac{1}{7} + \frac{1}{11} - \frac{1}{15} + \frac{1}{19} - \ldots = \frac{\pi}{4\sqrt{2}} - \frac{1}{2\sqrt{2}} \log (1+\sqrt{2}).$$

(ii) Consider the equality.

$$\frac{1}{a} - \frac{1}{a+b} + \frac{1}{a+2b} - \frac{1}{a+3b} + \ldots = \int_0^1 \frac{x^{a-1}}{1+x^b}\, dx \quad (a, b, >0)$$

Here by giving particular values to a and b we can get the sum of not only the above series, but also of many others of its type.

The given series corresponds to the values $a=3$ and $b=4$.

$$\text{Hence } S = \int_0^1 \frac{x^2 dx}{1+x^4}$$

$$\text{i.e. } S = \frac{1}{4\sqrt{2}} \left[\log \frac{x^2 - x\sqrt{2}+1}{x^2 + x\sqrt{2}+1} \right]_0^1 + \frac{1}{2\sqrt{2}} \left[\tan^{-1}(x\sqrt{2}+1) + \tan^{-1}(x\sqrt{2}-1) \right]$$

$$= \frac{1}{4\sqrt{2}} \log \frac{2-\sqrt{2}}{2+\sqrt{2}} + \frac{1}{2\sqrt{2}} \left[\tan^{-1}(\sqrt{2}+1) + \tan^{-1}(\sqrt{2}-1) \right]$$

$$= \frac{1}{4\sqrt{2}} \log \frac{\sqrt{2}-1}{\sqrt{2}+1} + \frac{\pi}{4\sqrt{2}}$$

$$= \frac{\pi}{4\sqrt{2}} - \frac{1}{2\sqrt{2}} \log (\sqrt{2}+1)$$

Question 921.

(H. R. KAPADIA) :—Prove that the infinite product

$$(1+x^{2n})\left(1+\frac{x^{2n}}{3^{2n}}\right)\left(1+\frac{x^{2n}}{5^{2n}}\right) \quad \ldots \quad \ldots$$

$$=\frac{1}{2^{\frac{n}{2}}}\prod_{1}^{n-1}(\cosh \pi x\alpha_r+\cos\pi x\beta_r)$$

$$\text{or} = \frac{1}{2^{\frac{1}{2}(n-1)}}\cosh\frac{\pi x}{2}\prod_{1}^{n-1}(\cosh \pi x\alpha_r+\cos\pi x\beta_r).$$

according as n is even or odd, if α_r, β_r denote $\sin\frac{r\pi}{2n}$ and $\cos\frac{r\pi}{2n}$ respectively, where r is an odd number.

Solution by Hemraj and M. K. Kewalramani.

Consider the equation $\frac{x^{2n}}{k^{2n}}+1=0$; clearly $\frac{x}{k}=\cos\frac{2\rho+1}{2n}\pi+i\sin\frac{2\rho+1}{2n}\pi$ where $\rho=0, 1, 2 \ \ldots \ \ldots \ (2n-1)$

i.e. if r is odd $=\cos.\frac{r\pi}{2n}+i\sin\frac{r\pi}{2n}=\beta_r+i\alpha_r$

If one root is w, the other must be $-w$, the third w^{-1}, and the fourth $(-w^{-1})$,—provided n is even. But if n is odd, there are two roots $\pm i$ corresponding to $\rho=\frac{n-1}{2}$ and $\rho=n+\frac{n-1}{2}$; and the other $(2n-2)$ roots can be divided into fours. Also it is evident that if any other factor be equated to zero, the roots will depend on k and w's must be the same.

Hence $$\frac{x^{2n}}{k^{2n}}+1=\prod_{\rho=0}^{n-1}\left(\frac{x^2}{k^2}-\omega^2{}_\rho\right)=(-1)^n\prod_{0}^{n-1}(\omega\rho)^2$$

$$\times\prod_{0}^{n-1}\left(1-\frac{x^2}{k^2\rho^2}\right)=\prod_{0}^{n-1}\left(1-\frac{x^2}{k^2\omega_\rho{}^2}\right)$$

$\therefore \prod(\omega_\rho{}^2)=\pm 1$ according as n is even or rodd.

Hence $\prod_{r=1}^{\infty}\left(1+\frac{x^{2n}}{k^{2n}}\right)=\prod_{k=1}^{\infty}\left[\prod_{\rho=0}^{n-1}\left(-\frac{x^2}{k^2\omega^2{}_{\rho}}\right)\right]$

$$=\prod_{0}^{n-1}\left[\prod_{0}^{\infty}\left(1-\frac{x^2}{k^2\omega_{\rho}{}^2}\right)\right]$$

$$=\prod_{0}^{n-1}\left(\cos\frac{x}{2\omega_{\rho}}\right) \qquad \therefore k \text{ is odd} \qquad \ldots \qquad \ldots \qquad \textbf{(A)}$$

$$=\prod_{0}^{2n-1}\left\{\cos\frac{\pi x}{2}(\beta_r-i\alpha_r)\right\} \qquad \therefore r=2\rho+1$$

$$=\prod_{1}^{2n-1}(\cos\tfrac{1}{2}\pi x\,\beta_r\cosh\tfrac{1}{2}\pi x\,\alpha_r+i\sin\tfrac{1}{2}\pi x\,\beta_r\sinh\tfrac{1}{2}\pi x\,\alpha_r)$$

$$=\prod_{1}^{n-1}(\cos^2\tfrac{1}{2}\pi x\,\beta_r\cosh^2\tfrac{1}{2}\pi x\,\alpha_r+\sin^2\tfrac{1}{2}\pi x\,\beta_r\sinh^2\tfrac{1}{2}\pi x\,\alpha_r)$$

(n even)

$$\text{or}=\cosh\tfrac{1}{2}\pi x\prod_{1}^{n-2}(\cos^2\tfrac{1}{2}\pi x\,\beta_r\cosh^2\tfrac{1}{2}\pi x\,\alpha_r+\sin^2\pi x\,\beta_r\sinh^2\tfrac{1}{2}\pi x\alpha_r)$$

(n odd)

[Notice that $\cosh\frac{1}{2}\pi x$ corresponds to $\omega_{\rho}=i$ and that if one factor in (A) is $\cos\frac{\pi x}{2\omega_{\rho}}$, the other must be $\cos\frac{\pi x}{2}\omega_{\rho}$]

$$=\frac{1}{2^{\frac{1}{2}n}}\prod_{1}^{n-1}(\cosh\pi x\alpha_r+\cos\pi x\,\beta_r)\ \text{or}=$$

$$\frac{\cosh\frac{1}{2}\pi x}{2^{\frac{1}{2}(n-1)}}\prod_{1}^{n-1}(\cosh\pi x\,\alpha_r+\cos\pi x\,\beta_r)$$

Note.—This question was originally constructed by Glaisher. See. Hobson's Trigonometry. Q. 32 Ch. XVII.

5

Question 924.

(S. Malhari Rao.):—Complete the following pandiagonqual Bears.

24				
	30			
		1		
			7	
				13

Solution by R. J. Pocock and Sadanand

The numbers given may be written

$$20+4,\ 25+5,\ 0+1,\ 5+2,\ 10+3.$$

Consider the following squares A, B and C. If each number in A is multiplied by 5 and added to the corresponding number in as we get the required pandiagonal magic square C.

A

4	0	2	5	1
2	5	1	4	0
1	4	0	2	5
0	2	5	1	4
5	1	4	0	2

B

4	3	2	1	5
1	5	4	3	2
3	2	1	5	4
5	4	3	2	1
2	1	5	4	3

C

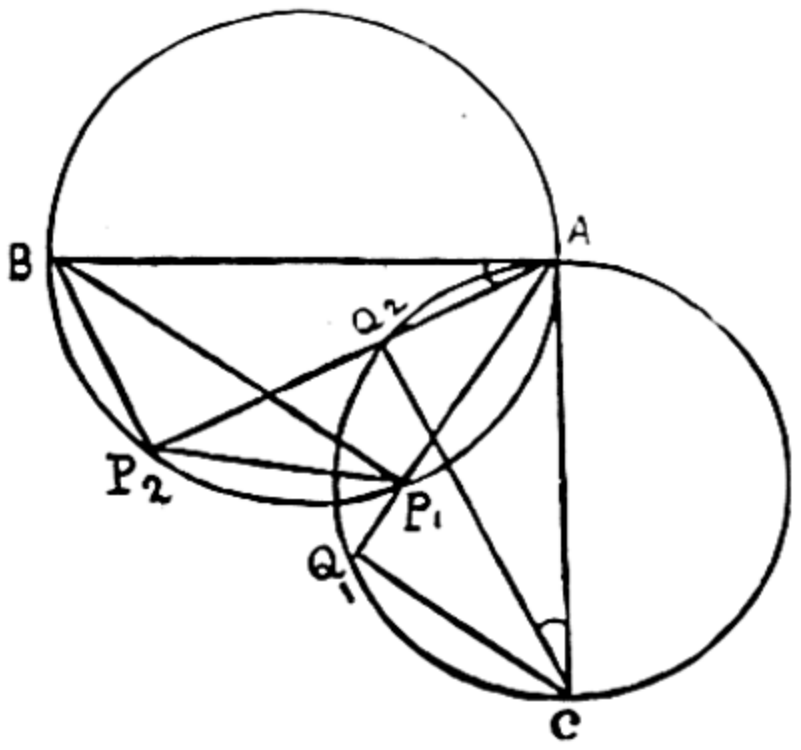
24	3	12	26	10
11	30	9	23	2
8	22	1	15	29
5	14	28	7	21
27	6	25	4	13

Question 926.

(S. R. RANGANATHAN):—If AB, AC be two straight lines intersecting at right-angles P_1, P_2 any two poiuts on the circle on A B as diameter and Q_1, Q_2 the points of intersection of P_1 A and P_2 A respectively with the circle on AC as diameter, show that P_1P_2 and Q_1Q_2 are at right angles to each other.

Solution by R.J. Pocock, P.M.H. Baker, C. Krishnamachary, R. Srinivasan K.B. Madhava, P.R. Venkatakrishnaiya, Hemraj and many others.

In the circle AP_1P_2B $\angle P_2AB = \angle P_2P_1B$.



Also $P_2AB = AQ_1Q_2$ since BA touches the circle ACQ_1Q_2.

$$\therefore\ P_2P_1Q_1 + P_1Q_1Q_2 = BP_1Q_1 = 90^\circ.$$

Hence P_1P_2 and Q_1Q_2 are at right angles.

[An obvious extension of the Question is the following :—

"Any two circles intersect at A and A P_1Q_1, A P_2Q_2 are drawn to cut them again in (P_1,Q_1), (P_2,Q_2), Shew that the angle between P_1P_2 and Q_1Q_2 is equal to the angle between the circles."]

Questions 942 and 943.

942. (HEMRAJ):—Show that if p be prime

$$\left[\sum_{1}^{p-1}\left(\frac{1}{n}\right) \div \frac{p^2}{p!}\right] \text{ is an integer}$$

943. (HEMRAJ) :—If x be prime and

$$(x+1)\,(x+2)\ldots\ldots(x+n-1)$$
$$\equiv x^{n-1}+A_1\,x^{n-2}+A_2\,x^{n-3}+\ldots A_{n-1}$$

shew that all the odd A's except A_1 are divisible n^2.

Solution and Remarks by H. Br.

Writing $f(x)$ for the given expression, we have

$$\begin{aligned}
f(-x) &= (-x+1)\,(-x+2)\ldots\ldots(-x+n-1)\\
&= (x-1)\,(x-2)\ldots\ldots(x-n+1)\\
&= (x-n+n-1)\,(x-n+n-2)\ldots(x-n+1)\\
&= f(x-n)\\
&= f(x)-n\,|f'(x)+\frac{n^2}{2}f''(x)-\ldots\ldots\\
&= f(x)-n\,\{(n-1)\,x^{n-2}+(n-2)\,|A_1\,x^{n-3}+\ldots\ldots\}+\ldots\ldots\\
&= f(x)-n\,(n-1)\,x^{n-2}+n^2\,F(x),
\end{aligned}$$

where F (x) is a polynomial in x with integral co-efficients. [This follows from the fact that the A's, upto A_{n-2} are multiples of n.]

Hence

$$f(x)-f(-x)\equiv n\,(n-1)\,x^{n-2}+0\ (\text{mod},\ n^2).$$

This being an identity, we can equate co-efficients of powers of x. We get

$$A_1=\tfrac{1}{2}\,n\,(n-1)\,;\ A_3\equiv A_5\equiv\ldots\ldots\equiv A_{n-2}\equiv 0\ (\text{mod},\ n^2).$$

Note : By considering terms involving n^2 and n^3, we get

$$2\,A_k\equiv n\,(n-k)\,A_{k-1}$$

where k is odd and >3. This is easily proved.

Remarks on Q. 943 by Mr. N. G. Leathar St. Stephen's College, Delhi.

This problem was set by me in the Punjab University M. A. Exam. of this year in the following form :—

By considering the equation $(x-1)(x-2)\ldots(x-p+1)$

$$\equiv x^{p-1}-C_1x^{p-2}+C_2x^{p-3}+\ldots(-)^{p-1}(p-1)\,!=0$$

shew from the results of Qn. IV that, if p be a prime all the coefficients $C_1, C_2\ldots$are divisible by p, and that the odd coefficients $C_3, C_5,\ldots$are divisible by p^2.

This was set as a rider to Newton's relations between the sums of the powers of the roots and the coefficients of an algebraical equation.

The reference to Qn. IV is to the theorem that :—

If S_r denote the sum of the r^{th} powers of the first $(p-1)$ natural numbers, $S_{2r+1}=S_1^2 . F(S_1)$ and $S_{2r}=S_2 G(S_1)$, where F and G are polynomials of degree $(r-1)$.

Newton's relations give

$$rC_r-S_1C_{r-1}+S_2C_{r-2}-S_3C_{r-3}+\ldots+(-)^rS_r=0 \quad \ldots \quad \ldots \quad (A)$$
$$(r=1, 2, \ldots, p-1)$$

I. $S_1, S_2, \ldots S_r$ are all divisible by p (assumed); therefore since r is prime to p, C_r is divisible by p, for all values of r from 1 to $(p-1)$.

II. If r is odd and $\not< 3$, S_r is divisible by p^2,

$S_1, S_2 \ldots S_{r-1}$ are all divisible by p,

and $C_{r-1}, C_{r-2} \ldots C_1$ are also all divisible by p.

Thus every term of (A), except the first, is divisible by p^2. Also r is prime to p.

Hence C_r is divisible by p^2, if r is odd and $\not< 3$

Question 944.

(SADANAND) :—Prove the identity

$$\phi(x)+S_1\phi'(x).\frac{x}{1!}-S_3\phi'''(x).\frac{x^3}{3!}+\ldots\ldots$$

$$\equiv\phi(2x)-S_2\phi''(2x)\frac{x^2}{2!}+S_4\phi''(2x).\frac{x^4}{4!}\ldots\ldots$$

where S_{2n} is the n^{th} Eulerian number and S_{2n-1} is the n^{th} "prepared Bernoullian" number.

Deduce that $\log 2=\frac{S_1}{1}-\frac{S}{2^3.1}-\frac{S_3}{3}+\frac{S_4}{2^5.2}+\frac{S_5}{5}-\ldots\ldots$

Solution by A. C. L. Wilkinson.

In the identity $\frac{2e^x}{e^x+e^{-x}}=1+S_1x-S_3\frac{x^3}{3!}+S_5\frac{x^5}{5!}-\ldots$

where $S_1, S_3, S_5, \ldots$ are the "prepared Bernoullians," put $\lambda\frac{d}{dx}$ for x and operate with both sides on $\phi(x)$, we get

$$\phi(x)+\lambda S_1\phi'(x)-\frac{\lambda^3}{3!}S_3\phi'''(x)+\ldots = \frac{2}{e^{\lambda\frac{d}{dx}}+e^{-\lambda\frac{d}{dx}}}\phi(x)$$

$$\frac{2}{e^{\lambda\frac{d}{dx}}+e^{-\lambda\frac{d}{dx}}}\phi(x+\lambda)$$

$$= \phi(x+\lambda) - \frac{\lambda^2}{2!} S_2\phi''(x+\lambda) + \frac{\lambda^4}{4!_1} S_4\phi''''(x+\lambda) - \ldots$$

Put $\lambda = x$ and we have the result given.

If we put $\phi(x) = \log x$ and then put $\lambda = 1$, we get log 2 in the form given which is however a divergent series.

Question 949.

(C. KRISHNAMACHARY):—With the usual notation for the numbers of Bernoulli and Euler, show that

$$2n = B_1 \cdot 2^2 (2^2-1) \binom{2n}{1} - B_2\, 2^4 (2^4-1) \binom{2n}{4} + B_3\, 2^6 (2^6-1) \binom{2n}{6} - \ldots + (-1)^{n-1} B_n\, 2^{2n} (2^{2n}-1)$$

$$(2n+1)\, E_n = B_n\, 2^{2n} (2^{2n}-1) \binom{2n+1}{1} - B_{n-1}\, 2^{2n-2} (2^{2n-2}-1) \binom{2n+1}{3} + \ldots + (-1)^{n-1} B_1\, 2^2 (2^2-1) \binom{2n+1}{2} + (-1)^n (2n+1).$$

Solution by A. C. L. Wilkinson.

The first result is proved in Edwards: *Differential Calculus*, second edition, *p*.502. It can also be proved as follows:

$$\frac{2x}{e^{2x}-1} = 1 - n + \Sigma(-)^{n-1} \frac{2^{2n} B_n x^{2n}}{(2n)}$$

$$\frac{4x}{e^{4x}-1} = 1 - 2x + \Sigma(-)^{n-1} \cdot \frac{2^{4n} B_n x^{2n}}{(2n)};$$

whence subtracting

$$\frac{2x}{e^{2x}+1} = x + \Sigma(-)^n \frac{2^{2n}(2^{2n}-1) - B_n x^{2n}}{(2n)!}$$

or $$e^{-x} = \left\{ 1 + \Sigma(-)^n 2^{2n} \frac{(2^{2n}-1)\, B_n x^{2n-1}}{2n!} \right\} \times \left\{ 1 + \Sigma \frac{x^{2n}}{2n!} \right\}.$$

Equating the coefficients of x^{2n-1}, we obtain the first result.

Again, in

$$\frac{2e^x}{e^{2x}+1} = \operatorname{sech} x = 1 + \Sigma(-)^n E_n \frac{x^{2n}}{(2n)!}$$

$$= \left\{ 1 + \Sigma(-)^n B_n \frac{2^{2n}(2n-1)}{(2n)!} x^{2n-1} \right\} \times \left\{ 1 + \Sigma \frac{x^n}{n!} \right\},$$

equating the coefficient of x^{2n} on both sides, we obtain the second result.

QUESTIONS FOR SOLUTION.

984. (S. R. RANGANATHAN) :—A prism has for its cross section a quadrilateral ABCD with $B=75°$, $C=105°$ and $D=90°$ and $\sqrt{3}\,AB=\sqrt{2}\,BC$. Show that a ray of light incident on the face AB at the angle $\sin^{-1}\frac{M}{2}$ and emerging out after once being totally reflected would undergo a deviation of 90°, M being the coefficient of refraction of the substance of the prism.

985. (M. K. KEWALRAMANI) :—If A′, B′, C′ are points in the sides of the triangle ABC, such that

$$BA' : A'C = CB' : B'A = AC' : C'B = M : 1,$$

find an expression for the radius of the circumcircle of the triangle XYZ formed by the intersections of AA′, BB′, CC′.

986. (M. K. KEWALRAMANI) :—At any point P of a curve equal arcs PQ, PR are measured in the same direction along the curve and along the parabola of closest contact. Prove that ultimately

$$QR=\left\{3+\tfrac{1}{3}\left(\frac{d\rho}{ds}\right)^2-\rho\frac{d^2\rho}{ds^2}.\right\}\frac{(ds)^4}{24\rho^3}.$$

987. (V. TIRUVENKATACHARI) :—Show that

$$\int_0^{\frac{1}{2}\pi}\sin\theta(\log\cos\theta)d\theta=-\tfrac{1}{3}\left\{1+\frac{2}{5}+\frac{2.4}{5.7}+\frac{2.4.6}{5.7.9}+\ldots\right\}=-1.$$

988. (S. MALHARI RAO) :—Complete the following magic squares of 16 cells, by placing 8 other prime numbers in the vacant cells :

7	43	71	113
73	97	23	41

989. (S. MALHARI RAO) :—Give three solutions in positive integers of the equation

$$(x+y+z)^2=xyz.$$

990. (S. MALHARI RAO) :—Give a solution in positive integers of

$$(x_1+x_2+\ldots x_n)^{\frac{1}{2}(n-1)}=x_1.x_2.x_3\ldots x_n$$

when n is odd ; and derive a method of solution of

$$(x_1+x_2+\ldots x_n)^{\frac{1}{2}n}=x_1x_2\ldots x_n$$

when n is even and greater than 2.

Hence deduce an expression for any fraction $\left(\frac{1}{n}\right)$ as the sum of a series of r fractions

$$\frac{1}{x_1}+\frac{1}{x_2}+\ldots\frac{1}{x_r}.$$

991. (S. NARAYANAN) :—ABC is a triangle whose incentre is I and orthocentre H. If the incentre to the ex-medial triangle of ABI be K and KH bisected at L, prove that the nine-points centre of ABC is the middle point of LI.

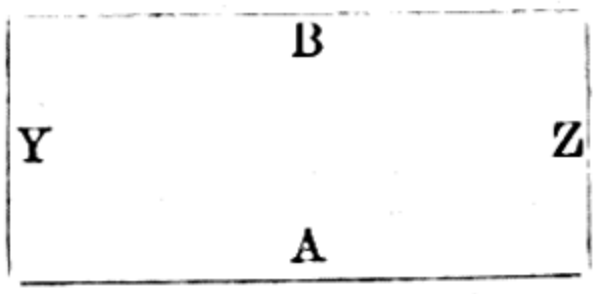
992. (S. NARAYANAN):—The internal bisectors of the angles A, B,C of a triangle meet the sides of the medial triangle DEF at P, Q, R. Prove that the conic which touches these sides at P, Q, R touches also the principal Feuerbach tangent of DEF.

993 (COMMUNICATED BY PROF HEMRAJ) :—From point P perpendiculars PL, PM, PN are drawn to the sides of a triangle ABC. If AL, BM, CN meet at a point Q, find the locus of P and prove that PQ passes through a fixed point.

994. (K. B. MADHAVA) :—Conduct the following game of Bridge with Hearts as tramps in which A has to lead and A and B together have to make 7 tricks.

Hearts : Q. Diamonds : A ; 3 ; 2.
Clubs : 7 ; 6. Spades : K ; 8.

Hearts : 7 ; 6.
Diamonds : J ; 9 ; 8.
Clubs : Q ; J : 9.
Spades : None.



Hearts : 9, 8.
Diamonds : 10 ; 5.
Clubs : None.
Spades : Q ; 7 ; 6 ; 5.

Hearts : A ; K ; 5. Diamonds : K.
Clubs : A ; 10 ; 8 ; 4. Spades : None.

995. (T. P. TRIVEDI, M.A., LL.B.) :—Prove that

$$\int_0^1 x^{x^x} dx = \sum_{n=0}^{n=\infty} (-1)^n \left\{ \frac{1}{1^{n+1}} - n.\frac{n+1}{2^{n+2}} + \frac{n^2}{2!}\frac{(n+1)(n+2)}{3^{n+1}} - \frac{n^3}{3!}\frac{(n+1)(n+2)(n+3)}{4^{n+4}} + \cdots \right\}$$

996. (T. P. TRIVEDI, M.A, LL. B.) :—Find the conditions that the conics

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

and

$$a'x^2+2h'xy+b'y^2+2g'x+2f'y+c'=0$$

have one asymptote in common..

997. (C. KRISHNAMACHARY) :—A uniform chain in partly coiled on a table with one end just carried over a smooth pulley at a height $2h$ vertically above the coil, and to this end is attached a mass M. Discuss the motion, shewing that the system will come to rest when M reaches the table provided its weight is nearly equal to that of a length h of the chain.

List of Periodicals Received.

(*From 16th July to* 15*th September* 1918).

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1. American Mathematical Monthly, Vol. 25, No. 4 & 5 April and May 1918.

2. Mathematical Questions and Solutions, Vol. 5, No. 4, April 1918 (5 Copies).

3. Mathematical Gazette, Vol. 9, No. 134, March 1918. (3 Copies).

4. Monthly Notices of the Royal Astronomical Society, Vol. 78, No. 5, March 1918.

5. Proceedings of the London Mathematical Society, Vol. 17, No. 1, June 1913.

6. School Science and Mathematics, Vol. 18, No. 5, May 1918 (2 Copies).

7. Tohoku Mathematical Journal, Vol. 13, No. 4,. June 1918.

Infinite Series and Arithmetical Functions.

By Prof. F. Hallberg.

(*Concluded from p.* 264, *Vol. X.*)

IV. Laguerre's Series—(*continued.*)

17. Before proceeding with our analysis of Laguerre's series, it will be well to examine on what foundations our theory rests, in order to distinguish between established and conjectured results.

The following two theorems given by Bromwich (*Infinite Series*, p. 171, ex. 15), are of importance for our purpose.

I. If $v_n(x)$ decreases as n increases and

$$\operatorname*{Lt}_{x\to 1} v_n(x)=1,$$

and if Σa_n is convergent (or divergent to $+\infty$), then

$$\operatorname*{Lt}_{x\to 1}\Sigma a_n v_n(x)=\Sigma a_n \text{ (or} +\infty).$$

II. If $A_n=\sum_{h=1}^{n} a_h$, $B_n=\sum_{h=1}^{n} b_h$,

and if

$$\operatorname*{Lt}_{n\to\infty} B_n/A_n=l,$$

then

$$\operatorname*{Lt}_{x\to 1}\left\{\sum b_n v_n(x)\right\}\Big/\left\{\sum a_n v_n(x)\right\}=l,$$

provided that A_n is always positive, and that A_n tends to ∞.

18. Put now

$$\sum_{n=1}^{\infty} f(n)\frac{x^n}{1-x^n}=\frac{x}{1-x}\sum_{n=1}^{\infty} a_n v_n(x) \quad \ldots \quad \ldots \quad (18.1)$$

Thus

$$a_n=\frac{f(n)}{n}, \quad \ldots \quad \ldots \quad \ldots \quad (18.11)$$

$$v_n(x)=\frac{nx^{n-1}}{1-x^n}(1-x)$$

$$=\frac{x^{n-1}+x^{n-1}+\ldots+x^{n-1}}{1\quad +x\quad +\ldots+x^{n-1}} \quad \ldots \quad \ldots \quad (18.12)$$

$$\therefore \quad \operatorname*{Lt}_{x\to 1} v_n(x)=1 \quad \ldots \quad \ldots \quad (18.2)$$

Also

$$v_n(x)-v_{n+1}(x)=$$

$$=\frac{(x^{n-1}-x^n)+(x^{n-1}-x^{n+1})+\ldots+(x^{n-1}-x^{2n-1})}{(1+x+\ldots+x^{n-1})(1+x+\ldots+x^{n-1}+x^n)}>0, \quad \ldots \quad (18.21)$$

if $0<x<1$.

$$\therefore \qquad v_n(x)>v_{n+1}(x) \quad \ldots \quad \ldots \quad \ldots \quad (18.22)$$

Thus the necessary conditions for $v_n(x)$ are satisfied, and we may apply Bromwich's theorems to Laguerre's series.

We have thus justified to some extent the use to which the series has already been put in the calculation of asymptotic mean values of arithmetical functions. By far the most difficult question, however—whether the asymptotic mean value really *exists*—cannot be solved in this way. But we have a powerful instrument for its calculation, provided its existence is *established* by other means, or *assumed*.

In general, our result, obtained by the aid of Cesaro's theorem, may be written:

$$\operatorname*{Lt}_{n\to\infty}\frac{1}{n}\sum_{h=1}^{n}\psi(h)=\sum_{n=1}^{\infty}\frac{f(n)}{(n)} \quad \ldots \quad \ldots \quad (18.3)$$

or

$$\frac{1}{n}\sum_{h=1}^{n}\psi(h)\sim\sum_{n=1}^{\infty}\frac{f(n)}{n}, \quad \ldots \quad \ldots \quad (18.4)$$

provided the limit exists (or tends to ∞) and the series converges (or diverges to ∞).

We will now consider some additional examples.

19. Using the notation of (11·21), we have

$$\sum_{n=1}^{\infty}\frac{f(n)}{n^{s-1}}\cdot\frac{x^n}{1-x^n}=\sum_{n=1}^{\infty}x^n\cdot\sum_{d:n}\frac{f(d)}{d^{s-1}} \quad \ldots \quad \ldots \quad (19\cdot1)$$

Thus (18·4) gives

$$\frac{1}{n}\sum_{h=1}^{n}\sum_{d:h}\frac{f(d)}{d^{s-1}}\sim\sum_{n=1}^{\infty}\frac{f(n)}{n^s}=\mathrm{F}(s), \quad \ldots \quad \ldots \quad (19\cdot11)$$

provided the *asymptotic mean value* (which will in the following be denoted A. M. V.) on the left exists and the Dirichlet's series converges.

Similarly

$$\sum_{n=1}^{\infty} \mu(n) \frac{f(n)}{n^{s-1}} \quad \frac{x^n}{1-x^n} = \sum_{n=1}^{\infty} x^n . \sum_{d\,;\,n} \mu(d) \frac{f(d)}{d^{s-1}}, \quad \ldots \quad (19{\cdot}2)$$

from which follows by (11·22), $f(n)$, satisfying the condition (11·2)

$$\frac{1}{n} \sum_{h=1}^{n} \sum_{d\,;\,h} \mu(d) \frac{f(d)}{d^{s-1}} \backsim \sum_{n=1}^{\infty} \mu(n) \frac{f(n)}{n^s} = \frac{1}{\mathrm{F}(s)}, \quad \ldots \quad (19{\cdot}21)$$

provided the A. M. V. exists and the Dirichlet's series converges.

The formulae (19·11), (19·21) may be regarded as generalisations of (13·24).—An interesting problem would be to examine, whether the A. M. V. and the series have the same convergence abscissa. This may be considered as very probable.

Using certain well-known formulae from the theory of Dirichlet's series*, we have thus in all probability :

For $f(n) = t(n)$:

$$\frac{1}{n} \sum_{h=1}^{n} \sum_{d\,:\,h} \frac{t(d)}{d^{s-1}} \backsim \zeta^2(s). \quad \ldots \quad \ldots \quad \ldots \quad (19{\cdot}12)$$

For $f(n) = \int(n)$:

$$\frac{1}{n} \sum_{h=1}^{n} \sum_{d\,:\,h} \frac{\int(d)}{d^{s-1}} \backsim \zeta(s).\, \zeta(s-1). \quad \ldots \quad \ldots \quad (19{\cdot}13)$$

For $f(n) = \phi(n)$:

$$\frac{1}{n} \sum_{h=1}^{n} \sum_{d\,:\,h} \frac{\phi(d)}{d^{s-1}} \backsim \frac{\zeta(s-1)}{\zeta(s)}, \quad \ldots \quad \ldots \quad (19{\cdot}14)$$

For $f(n) = (-1)^{n-1}$:

$$\frac{1}{n} \sum_{h=1}^{n} \sum_{d\,:\,h} \frac{(-1)^{d-1}}{d^{s-1}} \backsim (1-2^{1-s}).\, \zeta(s). \quad \ldots \quad (19{\cdot}15)$$

For $f(n) = 1$:

$$\frac{1}{n} \sum_{h=1}^{n} \sum_{d\,:\,h} \frac{1}{d^{s-1}} \backsim \zeta(s), \quad \ldots \quad \ldots \quad \ldots \quad (19{\cdot}16)$$

* See " Encyclopedie " T. I, Vol. 3, p. 268.

$$\frac{1}{n}\sum_{h=1}^{n}\sum_{d:h}\frac{\mu(d)}{d^{s-1}} \backsim \frac{1}{\zeta(s)}. \qquad \ldots \qquad \ldots \quad (19\,22)$$

Of these, (19·12), (19·16), (19·22) are probably valid for R $(s)>1$, (19·13), (19·14) for R $(s)>2$, (19·15) for R $(s)>0$.——Formula (19·16) is nothing else than (13·24).

20. Let $f(z)$ be any analytic function, and thus

$$f(z)-f(0)=\frac{z}{1}\cdot f'(0)+\frac{z^2}{2!}f''(0)+ \qquad \ldots \qquad \ldots \quad (20\cdot1)$$

Then

$$\sum_{n=1}^{\infty}\frac{z^n}{(n-1)!}f^{(n)}(0).\frac{xn}{1-x}=\sum_{n=1}^{\infty}x^n.\sum_{d:n}\frac{z^d}{(d-1)!}f^{(d)}(0), \quad \ldots \quad (20\cdot11)$$

from which follows

$$\frac{1}{n}\sum_{h=1}^{n}\sum_{d:h}\frac{z^d}{(d-1)!}f^{(d)}(0) \backsim f(z)-f(0), \qquad \ldots \quad (20\cdot12)$$

for a sufficiently small $|z|$, provided the A. M. V. exists.

In this way we obtain, for instance

$$\frac{1}{n}\sum_{h=1}^{n}\sum_{d:h}z^d \backsim -\log(1-z), \qquad \ldots \qquad \ldots \quad (20\cdot13)$$

when $|z|<1$.

For $z=-1$, the last formula becomes

$$\frac{1}{n}\sum_{h=1}^{n}\sum_{d:h}(-1)^{d-1} \backsim \log 2 \qquad \ldots \qquad \ldots \quad (20\cdot131)$$

Many more examples could of course be given, but would give us nothing fundamentally new.

It is further evident, that any other expansion of a function in the form of an infinite series will furnish similar results. To go into details would carry us to far. The following examples must suffice.

Formula (7.31), for instance, gives

$$\frac{1}{n}\sum_{h=1}^{n}\sum_{d:h}\frac{2dx^2}{x^2-d^2\pi^2} \backsim x\cot x-1, \qquad \ldots \qquad \ldots \quad (20\cdot2)$$

and the corresponding (7·33)

$$\frac{1}{n}\sum_{h=1}^{n}\sum_{d:\,h}\mu(d)\left[x\cot\frac{d}{x}-d\right]\sim\frac{2x^3}{x^2-\pi^2}, \qquad \ldots \quad (20{\cdot}21)$$

provided the limits on the left exist.

Similarly formula (11·31) gives

$$\frac{1}{n}\sum_{h=1}^{n}\sum_{d:\,h}d\left(\frac{s}{d}-\frac{s}{s+d}\right)\sim s(\psi(s)+C)+1, \qquad \ldots \quad (20{\cdot}3)$$

when s is not a negative integer, provided the A. M. V. exists; and (11·33) gives a corresponding formula.

21. Let p be a given prime number.

Define $f_p(n)=1$, if $n=p^r$, $r=$positive integer.
$f_p(n)=0$, if n contains other prime factors than p.
$f_p(1)=0$.
$\psi_p(n)=r$, if $n=p^r n'$, $n'\not\equiv 0 \pmod p$.
$\psi_p(n)=0$, if $n\not\equiv 0 \pmod p$.

Thus

$$\sum_{r=1}^{\infty}\frac{x^{p^r}}{1-x^{p^r}}=\sum_{n=1}^{\infty}\psi_p(n)\,x^n. \quad \ldots \qquad \ldots \quad (21.1)$$

From this we obtain in the usual manner

$$\frac{1}{n}\sum_{h=1}^{n}\psi_p(h)\sim\sum_{r=1}^{\infty}p^{-r}=\frac{1}{p-1}. \qquad \ldots \qquad \ldots \quad (21.11)$$

This result is capable of the following generalisation.

Let $(P)=(p_{n_1}, p_{n_2},\ldots, p_{n_k})$ be a given finite set of primes, k in number.

Define $f_p(n)=1$, if n is made up entirely of primes, belonging to the given set.

$f_p(n)=0$, if n contains other prime factors than those belonging to the given set.

$f_p(1)=0$.

and $\psi_p(n)=\prod_{i=1}^{k}(1+r_i)-1$, if $n=n'.\prod_{i=1}^{k}p_{n_i}^{r_i}$

$n'\not\equiv 0 \pmod{p_{n_i}};\ i=1, 2,\ldots, k).$

Thus

$$\sum_{n=1}^{\infty} f_p(n).\frac{x^n}{1-x^n}=\sum_{n=1}^{\infty}\psi_p(n).x^n, \qquad \ldots \qquad \ldots \quad (21.2)$$

from which follows

$$\frac{1}{n}\sum_{h=1}^{n}\psi_p(h) \backsim \sum\frac{1}{\pi_p-1} \qquad \ldots \qquad \ldots \qquad \ldots \quad (21.21)$$

where the sum to the right is extended over all numbers π_p, made up entirely of primes belonging to (P), and not forming an r^{th} power of any factor of $\left|\left|_{i=1}^{k} p_{n_i}\right.\right.$, r being ≥ 2.

22. Comparing (12·1), (12.11), (12·12) with the corresponding

$$\sum_{n=1}^{\infty} f_1(n)\ \frac{x^n}{1-x^n}=\sum_{n=1}^{\infty}x^n\,\psi_1(n), \qquad \ldots \qquad \ldots \quad (22.1)$$

where

$$\psi_1(n)=\sum_{d\,:\,n} f_1(d), \qquad \ldots \qquad \ldots \qquad \ldots \quad (22.11)$$

$$f_1(n)=\sum_{d\,:\,n}\mu(d)\psi_1\left(\frac{n}{d}\right), \qquad \ldots \qquad \ldots \quad (22\,12)$$

and where $f_1(n)$, and hence by (22.11) also $\psi_1(n)$, are *positive*, we obtain easily from Bromwich's theorem II and Cesaro's theorem of comparison for two divergent power series *, the following interesting result, provided

$$\sum_{n=1}^{\infty}\frac{f_1(n)}{n} \textit{ diverges, and both limits exist:}$$

$$\operatorname*{Lt}_{n\to\infty}\frac{\sum_{h=1}^{n}\frac{f(h)}{h}}{\sum_{h=1}^{n}\frac{f_1(h)}{h}}=\operatorname*{Lt}_{n\to\infty}\frac{\sum_{h=1}^{n}\psi(h)}{\sum_{h=1}^{n}\psi_1(h)} \qquad \ldots \qquad \ldots \quad (22.2)$$

(For the proof, use the value for $v_n(x)$ given in (18.12)).

* See Bromwich, *Infinite Series*, p. 131.

Ex 1. Let

$$\left.\begin{matrix} f(n)=\mu^2(n) \\ f_1(n)=1 \end{matrix}\right\} \quad \therefore \left.\begin{matrix} \psi(n)=p(n) \\ \psi_1(n)=t(n) \end{matrix}\right\}$$

Thus

$$\frac{\sum_{h=1}^{n} \frac{\mu^2(h)}{h}}{\sum_{h=1}^{n} \frac{1}{h}} \sim \frac{\sum_{h=1}^{n} p(h)}{\sum_{h=1}^{n} t(h)} \sim \frac{6}{\pi^2}$$

by (13.41), (13.12).

Hence

$$\sum_{h=1}^{n} \frac{\mu^2(h)}{h} \sim \frac{6}{\pi^2} \cdot \log n. \qquad \ldots \qquad \ldots \quad (22.21)$$

Ex. 2. Let

$$\left\{\begin{matrix} f(n)=\sum_{d:n} \mu(d) \cdot \phi\left(\frac{n}{d}\right) \\ f_1 n)=n \end{matrix}\right\} \left\{\begin{matrix} \therefore \psi(n)=\phi(n) \\ \therefore \psi_1(n)=\int(n) \end{matrix}\right\}$$

Thus

$$\frac{\sum_{h=1}^{n} \frac{1}{h} \cdot \sum_{d:h} \mu(d) \cdot \phi\left(\frac{h}{d}\right)}{\sum_{h=1}^{n} \frac{1}{h} \cdot h} \sim \frac{\sum_{h=1}^{n} \phi(h)}{\sum_{h=1}^{n} \int(h)} \sim \frac{36}{\pi^4}$$

by (13.32), (13.22)

Hence

$$\sum_{h=1}^{n} \frac{1}{h} \cdot \sum_{d:h} \mu(d) \ \phi\left(\frac{h}{d}\right) \sim \frac{36}{\pi^4} \cdot n. \qquad \ldots \quad (22.22.)$$

Ex. 3. Let

$$\left.\begin{matrix} f(n)=\phi(n) \\ f_1(n)=n \end{matrix}\right\} \quad \therefore \left.\begin{matrix} \psi(n)=n \\ \psi_1(n)= f(n) \end{matrix}\right\}$$

Thus

$$\frac{\sum_{h=1}^{n} \frac{\phi(h)}{h}}{\sum_{h=1}^{n} \frac{h}{h}} \geq \frac{\sum_{h=1}^{n} h}{\sum_{h=1}^{n} \int(h)} \geq \frac{6}{\pi^2},$$

by (13.22).

Hence

$$\sum_{h=1}^{n} \frac{\phi(h)}{h} \geq \frac{6}{\pi^2}. n. \quad \ldots \quad \ldots \quad \ldots (22\,23)$$

Putting $\psi(n) = \frac{\phi(n)}{n}$ in formula (12.1), we obtain from (18.3) by the aid of (22.23)

$$\sum_{n=1}^{\infty} \frac{1}{n} \sum_{d:n} \mu\left(\frac{n}{d}\right) \frac{\phi(d)}{d} =$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{d:n} \mu(d).\ d.\ \phi\left(\frac{n}{d}\right) = \frac{6}{\pi^2}. \quad \ldots (22.231)$$

We know, however, from (11.12), that

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^2} = \frac{1}{\zeta(2)} = \frac{6}{\pi^2},$$

from which we conclude

$$\sum_{d:n} \mu(d).\ d.\ \phi\left(\frac{n}{d}\right) = \mu(n), \quad \ldots \quad \ldots (22.3)$$

a formula, which is easily verified.

Formula (22.3) may be written

$$\frac{\mu(n)}{n} = \sum_{d:n} \mu(d) \frac{\phi\left(\frac{n}{d}\right)}{\frac{n}{d}},$$

from which follows, by (12.12)

$$\frac{\phi(n)}{n} = \sum_{d\,:\,n} \frac{\mu(d)}{d}. \quad \ldots \qquad \ldots \qquad \ldots \text{ (22.31)}$$

Various other general results similar to (22.2) may be obtained by using Frobenius's theorem with Bromwich's extension (See Bromwich, *Infinite Series*, pp. 132, 313).

23. *μ-inversion of Laguerre's series.*

Let in (12.1), $f(n)$ satisfy the condition (11.2), and μ-invert.

The result is

$$\sum_{r=1}^{\infty} \mu(r).\, f(r).\, \sum_{n=1}^{\infty} x^{nr}.\, \psi(n) = \frac{x}{1-x}. \quad \ldots \qquad \ldots \text{ (23.1)}$$

Hence:

$$\sum_{d\,:\,n} \mu(d).\, f(d).\, \psi\left(\frac{n}{d}\right) = 1, \qquad \ldots \qquad \ldots \text{ (23.11)}$$

provided always

$$f(m\ m') = f(m) f(m'). \qquad \ldots \qquad \ldots \qquad \ldots \text{ (23.111)}$$

Let now $\psi(n)$ satisfy (11.2), and μ—invert.

We obtain

$$\sum_{r=1}^{\infty} \mu(r).\, \psi(r).\, \sum_{n=1}^{\infty} \frac{x^{nr}}{1-x^{nr}} . f(n) = x$$

$$= \sum_{n=1}^{\infty} \frac{\mu(n) x^n}{1-x^n}, \text{ by (14.1).} \qquad \ldots \qquad \ldots \text{ (23.2)}$$

Hence:

$$\sum_{d\,:\,n} \mu(d).\, \psi(d).\, f\left(\frac{n}{d}\right) = \mu(n) \qquad \ldots \qquad \ldots \text{ (23.21)}$$

for $n > 1$, provided always

$$\psi(mm') = \psi(m)\psi(m') \qquad \ldots \qquad \ldots \text{ (23·211)}$$

We will consider a few simple examples.

Ex. 1. Put

$$f(n) = 1. \quad \therefore\ \psi(n) = t(n).$$

Hence by (23·11)

$$\sum_{d\,:\,n}\mu(d).\ t\left(\frac{n}{d}\right)=1=f(n),$$

which is evidently correct because of (12·12)

Ex. 2. Put

$$f(n)=n. \quad \therefore\ \psi(n)=\int(n)$$

Hence by (23.11).

$$\sum_{d\,:\,n}\mu(d).\ d.\ \int\left(\frac{n}{d}\right)=1\ .$$

The last formula may be written

$$\sum_{d\,:\,n}\mu(d).\ \frac{\int\left(\frac{n}{d}\right)}{\left(\frac{n}{d}\right)}\ =\ \frac{1}{n}\ ,$$

from which follows, by (12·12), (12· 11)

$$\frac{f(n)}{n}=\sum_{d\,:\,n}\frac{1}{d}=\sum_{d\,:\,n}\frac{d}{n}=\frac{1}{n}.\sum_{d\,:\,n}d,$$

an evident identity.

Ex. 3. Put

$$f(n)=\phi(n). \quad \therefore\ \psi(n)=n.$$

In this case, (23·21) becomes (22·3).

For examples of a more general nature, use for $f(n)$ or $\psi(n)$ the arithmetical functions defined in § 3. (towards the end of p. 175, Vol. IX).

V. Differentiation of Laguerre's Series.

24. The series (12.1) evidently admits of differentiation. The result may be written:

$$\sum_{n=1}^{\infty}\frac{f(n)}{n}.\frac{n^2.\ x^{n-1}}{(1+x+x^2+\ldots+x^{n-1})^2}=\frac{(1-x)^2}{x}.\sum_{n=1}^{\infty}n.\ \psi(n).\ x^n. \qquad (24.1)$$

If we here put

$$v'_n(x)=\frac{n^2.\ x^{n-1}(1-x)^2.}{(1-x^n)^2} \qquad \ldots \qquad \ldots \qquad \ldots(24.11)$$

we find that the condition

$$\lim_{x\to 1} v'_n(x)=1, \qquad \ldots \qquad \ldots \qquad \ldots \quad (24.12)$$

is satisfied.

Let $0<y<1$.

$$\therefore \quad 1+(1+y)+(1+y+y^2)+\ldots+(1+y+y^2+\ldots+y^{n-1})>$$
$$>y^{2n}.\ 1+y^{2n-1}.\ (1+y)+y^{2n-2}.\ (1+y+y^2)+\ldots$$
$$+y^{n+1}(1+y+y^2+\ldots+y^{n-1}).$$

Multiply by $(1-y)>0$.

$$\therefore \quad (1-y)+(1-y^2)+(1-y^3)+\ldots+(1-y^n)>$$
$$>y^{n+1}(1-y^n)+y^{n+2}(1-y^{n-1})+\ldots+y^{2n}(1-y).$$

$$\therefore \quad n+ny^{2n+1}>y(1+y+y^2+\ldots+y^{2n-1})=y.\ \frac{1-y^{2n}}{1-y}.$$

$$\therefore \quad n-ny+ny^{2n+1}-ny^{2n+2}>y-y^{2n+1}.$$

$$\therefore \quad \frac{n}{1-y^{2n}}>\frac{(n+1)y}{1-y^{2(n+1)}}.$$

Square, put $y=\sqrt{x}$, and multiply by $x^{n-1}(1-x)^2$, when we obtain

$$\frac{n^2x^{n-1}(1-x)^2}{(1-x^n)^2}>\frac{(n+1)^2x^n(1-x)^2}{(1-x^{n+1})^2},$$

or

$$v'_n(x)>v'_{n+1}(x) \qquad \ldots \qquad \ldots \qquad \ldots \qquad \ldots \quad (24.13)$$

for $0<x<1$.

We may therefore apply Bromwich's theorem I, on the left side of (24.1), using Cesaro's theorem on the right, when we obtain

$$\frac{2}{n^2}\sum_{h=1}^{n} h\,\psi(h)\sim\sum_{n=1}^{\infty}\frac{f(n)}{n}. \qquad \ldots \qquad \ldots \qquad \ldots \quad (24.2)$$

provided the limit exists (or tends to α), and the series converges (or diverges to ∞).

We may put (24.2) into a slightly different form.

For this purpose, assume $n\,\psi(n)=\theta(n)$;

$$\therefore \quad f(n)=\sum_{d:n}\mu(d)\,\psi\left(\frac{n}{d}\right)=\sum_{d:n}\mu\left(\frac{n}{d}\right)\psi(d)=$$
$$=\frac{1}{n}\sum_{d:n}\mu(d).\,d.\,\theta\left(\frac{n}{d}\right)=\sum_{d:n}\mu\left(\frac{n}{d}\right)\frac{\theta(d)}{d}.$$

Hence :

$$\frac{2}{n^2}\sum_{h=1}^{n}\theta\,(h) \backsim \sum_{n=1}^{\infty}\frac{1}{n^2}\sum_{d\,:\,n}\mu\,(d).\,d.\,\theta\left(\frac{n}{d}\right) =$$

$$= \sum_{n=1}^{\infty}\frac{1}{n}\sum_{d\,:\,n}\mu\left(\frac{n}{d}\right)\frac{\theta\,(d)}{d}, \qquad \ldots \qquad \ldots \quad (24{\cdot}21)$$

valid under similar conditions.

We will now consider some examples.

25. As in § 19., we have, putting $\frac{f\,(n)}{n^{s-1}}$ instead of $f\,(n)$ in (24·2)

$$\frac{2}{n^2}\sum_{h=1}^{n} h \sum_{d\,:\,h}\frac{f\,(d)}{d^{s-1}} \backsim \mathrm{F}\,(s), \quad \ldots \qquad \ldots \qquad \ldots \qquad \ldots \quad (25{\cdot}1)$$

and similarly, *if the condition* (11· 2) *is satisfied* :

$$\frac{2}{n^2}\sum_{h=1}^{n} h \sum_{d\,:\,h}\mu\,(d)\frac{f\,(d)}{d^{s-1}} \backsim \frac{1}{\mathrm{F}\,(s)}, \qquad \ldots \qquad \ldots \qquad \ldots \quad (25{\cdot}2)$$

always provided that the limits on the left exists and that the Dirichlet's series on the right converges.

Ex. : For $f\,(n)=1$, formula (25.1) gives

$$\frac{2}{n^2}\sum_{h=1}^{n} h^{2-s} \sum_{d\,\cdot\,h} d^{s-1} \backsim \zeta\,(s), \qquad \ldots \qquad \ldots \qquad \ldots \quad (25.11)$$

probably valid for $\mathrm{R}\,(s)>1$.

The case $s=2$ is particularly important. We obtain

$$\sum_{h=1}^{\infty}\int(h) \backsim \frac{\pi^2}{12}n^2, \ldots \qquad \ldots \qquad \ldots \qquad \ldots \quad (25.111)$$

in agreement with Dirichlet's result, given in formula (13.22). Hence the alternative result (see p. 257 of the present paper) is incorrect.

The case $s=1$ shows that $\sum_{h=1}^{n} h.\,t\,(h)$ is of an order greater than $n.^2$

For $f(n)=1$, formula (25.2) gives

$$\frac{2}{n^2}\sum_{h=1}^{n}{}' h^{2-s}\sum_{d\,:\,h}{}' \mu\left(\frac{h}{d}\right) d^{s-1} \sim \frac{1}{\zeta(s)}, \qquad \ldots \qquad \ldots \quad (25.1)$$

which for $s=2$ becomes

$$\sum_{h=1}^{\infty} \phi(h) \sim \frac{3}{\pi^2} n^2 \qquad \ldots \qquad \ldots \qquad \ldots \quad (25.21)$$

in agreement with (13.32).

Other special results may easily be obtained in a way similar to that of § 19

The matter of § § 20, 21, may also be easily extended on the present lines.

26. It is further evident that we may, by the aid of Bromwich's theorem II, obtain a result corresponding to (22·2). Under the same conditions as in § 22, we have :

$$\operatorname*{Lt}_{n\to\infty} \frac{\sum_{h=1}^{n}{}' \frac{f(h)}{h}}{\sum_{h=1}^{n}{}' \frac{f_1(h)}{h}} = \operatorname*{Lt}_{n\to\infty} \frac{\sum_{h=1}^{n}{}' h\,\psi(h)}{\sum_{h=1}^{n}{}' h\,\psi_1(h)} \qquad \ldots \qquad \ldots \quad (26\cdot1)$$

From this result, new asymptotic formula may be obtained

We note (see § 22., Ex. 3.) $$\frac{\sum_{h=1}^{n}{}' h\int(h)}{\sum_{h=1}^{n}{}' h^2} \sim \frac{\pi^2}{6},$$

But

$$\sum_{h=1}^{n}{}' h^2 = \tfrac{1}{6} n(n+1)(2n+1) \sim \frac{n^3}{3}$$

Hence

$$\sum_{h=1}^{n} h\int(h) \sim \frac{\pi^2}{18} n^3. \qquad \ldots \qquad \ldots \qquad \ldots \quad (26{\cdot}2)$$

27. We may apply μ inversion to formula (24.1) written in the form

$$\sum_{n=1}^{\infty} nf(n)\frac{x^n}{(1-x^n)^2} = \sum_{n=1}^{\infty} n\psi(n)x^n, \qquad \ldots \qquad \ldots \quad (27.1)$$

but the results are the same as those of § 23.

We will consider specially the formula (14.22) (where the brackets in the denominator of the fraction under the sum should be removed), comparing it with the corresponding

$$\sum_{n=1}^{\infty} \frac{n\ddagger(n)x^n}{(1-x^n)^2} = \frac{x(1+x)}{(1-x)^3}. \qquad \ldots \qquad \ldots \quad (27.2)$$

We obtain

$$\frac{\sum_{n=1}^{\infty} \frac{\phi(n)}{n} v_n(x)}{\sum_{n=1}^{\infty} \frac{\phi'(n)}{n} v_n'(x)} = \frac{1}{1+x}, \qquad \ldots \qquad \ldots \quad (27.21)$$

and hence

$$\operatorname*{Lt}_{x\to 1} \frac{\sum_{n=1}^{\infty} \frac{\phi(n)}{n} v_n(x)}{\sum_{n=1}^{\infty} \frac{\phi'(n)}{n} v_n'(x)} = \frac{1}{2}, \qquad \ldots \qquad \ldots \quad (27.3)$$

This is an interesting relation between two divergent series, corresponding terms of which tend to equality. The anomaly is of course due to the fact, that $v_n(x)$ and $v_n'(x)$ approach unity in a different manner.

VI. Integration of Laguerre's Series.

28. Dividing both sides of (12.1) by x and integrating, we obtain

$$\sum_{n=1}^{\infty} \frac{f(n)}{n} \cdot \log(1-x^n) = -\sum_{n=1}^{\infty} \frac{\psi(n)}{n} \cdot x^n, \quad \ldots \quad \ldots \quad (28.1)$$

or

$$\prod_{n=1}^{\infty} (1-x^n)^{\frac{f(n)}{n}} = e^{-\sum_{n=1}^{\infty} \frac{\psi(n)}{n} \cdot x^n} \quad \ldots \quad \ldots \quad (28.11)$$

no constant of integration being required.

From this result we may deduce many interesting identities.

We note:

$$\log \prod_{n=1}^{\infty} (1-x^n) = -\sum_{n=1}^{\infty} \frac{f(n)}{n} x^n. \quad \ldots \quad \ldots \quad (28.2)$$

$$\log \prod_{n=1}^{\infty} (1-x^n)^{\frac{1}{n}} = -\sum_{n=1}^{\infty} \frac{t(n)}{n} x^n. \quad \ldots \quad \ldots \quad (28.3)$$

$$\log \prod_{n=1}^{\infty} (1-x^n)^{\frac{\mu(n)}{n}} = -x. \quad \ldots \quad \ldots \quad \ldots \quad (28.4)$$

$$\log \prod_{n=1}^{\infty} (1-x^n)^{\frac{\phi(n)}{n}} = -\frac{x}{1-x}. \quad \ldots \quad \ldots \quad \ldots \quad (28.5)$$

29. Formula (28.1) may be written

$$\ldots \quad \frac{\sum_{n=1} \frac{\cdot\, n)}{n} \cdot x^n}{\sum_{n=1}^{\infty} \frac{1}{n} \cdot x^n}$$

$$= \sum_{n=1}^{\infty} \frac{f(n)}{n} + \frac{\sum_{n=1}^{\infty} \frac{f(n) \log}{n} \cdot \frac{\log(1+x+\ldots+x^{n-1})}{\log n}}{\log(1-x)} \quad \ldots \quad (29.1)$$

Put

$$v''_n(x)=\frac{\log(1+x+\ldots+x^{n-1})}{\log n} \quad \ldots \quad \ldots \quad (29.11)$$

$$\therefore \quad \operatorname*{Lt}_{x\to 1} v''_n(x)=1. \quad \ldots \quad \ldots \quad (29.111)$$

Also

$$v''_n(x)>v''_{n+1}(x), \quad \ldots \quad \ldots \quad (29.112)$$

since the fraction $v''_n(x)$ is<1 for $x<1$, and the numerator is increased by a smaller quantity than the denominator, when $v''_n(x)$ is changed into $v''_{n+1}(x)$.

Hence Bromwich's theorem I, may be applied to the series in the numerator of the last term in (29.1), so that, in particular this series approaches

$$\sum_{n=1}^{\infty}\frac{f(n)\log n}{n},$$

if the latter series converges.

Hence we derive the following theorem, by the aid of Cesaro's theorem :

$$\frac{1}{\log n}\sum_{h=1}^{n}\frac{\psi(n)}{n}\backsim\sum_{n=1}^{\infty}\frac{f(n)}{n}, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (29.2)$$

provided the limit exists, and the series

$$\sum_{n=1}^{\infty}\frac{f(n)\log n}{n}$$

is convergent.

(This is of course only a particular case of a more general theorem, where the condition for the validity of (29.2) is that the last term in (29.1) should approach zero, as x tends to unity).

30. As an example of the last theorem, substitute in (29.2) $\frac{f(n)}{n^{s-1}}$ for $f(n)$

We deduce, as in §§ 19., 25.,

$$\frac{1}{\log n}\sum_{h=1}^{n}\frac{1}{h}\sum_{d\,:\,h}\frac{f(d)}{d^{s-1}}\backsim F(s), \quad \ldots \quad \ldots \quad (30{\cdot}1)$$

provided the limit on the left exists and the Dirichlet's series converges,

The last condition is obtained from the theorem • that if $F(s)$ converges for $s=s_o$,

$$F'(s) = -\sum_{n=1}^{\infty} \frac{f(n)\log n}{n^s} \qquad \ldots \qquad \ldots \quad (30{\cdot}11)$$

converges for $s>s_o$. Hence, when $R(s)$ is equal to the convergence-abscissa of $F(s)$, a special examination is necessary.

Ex. For $f(n)=1$, formula (30·1) gives

$$\frac{1}{\log n} \sum_{h=1}^{n} \frac{1}{h^s} \sum_{d\,:\,h} d^{s-1}\ \zeta(s), \qquad \ldots \qquad \ldots \quad (30\,12)$$

which for $s=2$ becomes

$$\sum_{h=1}^{n} \frac{f(h)}{h} \sim \frac{\pi^2}{6} \log n \qquad \ldots \qquad \ldots \qquad \ldots \quad (30{\cdot}12)$$

Under conditions similar to the above we obtain in addition

$$\frac{1}{\log n} \sum_{h=1}^{n} \frac{1}{h} \sum_{d\,:\,h}^{n} \mu(d) \frac{f(d)}{d^{s-1}} \sim \frac{1}{F(s)}, \qquad \ldots \qquad \ldots \quad (30{\cdot}2)$$

provided the condition (11.2) *is satisfied.*

Ex. For $f(n)=1$, formula (30·2) becomes

$$\frac{1}{\log n} \sum_{h=1}^{n} \frac{1}{h^s} \sum_{d\,:\,h}^{n} \mu\left(\frac{h}{d}\right) d^{s-1} \sim \frac{1}{\zeta(s)} \qquad \ldots \qquad \ldots \quad (30{\cdot}21)$$

which for $s=2$ gives

$$\sum_{h=1} \frac{\phi(h)}{h^2} \sim \frac{6}{\pi^2} \log n \qquad \ldots \qquad \ldots \qquad \ldots \quad (30{\cdot}212)$$

31. From (28·1) we deduce, putting $f(n)=\frac{1}{n}$,

$$-\frac{\pi^2}{6} \log(1-x) - \sum_{n=1}^{\infty} \frac{1}{n} x^n \sum_{d\,:\,n} \frac{1}{d} =$$

• *See* Landau : *Handbuch der Lehre von der Verteilung der Primzahlen*, pp. 108, 109.

$$= \sum_{n=1}^{\infty} \frac{x^n}{n}\left(\frac{\pi^2}{6} - \frac{f(n)}{n}\right)$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^2} \log\,(1+x+\ldots+x^{n-1}). \qquad \ldots \quad (31.1)$$

Hence:

$$\sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{\pi^2}{6} - \frac{f(n)}{n}\right) = \sum_{n=1}^{\infty} \frac{\log n}{n^2}, \quad \ldots \qquad \ldots \quad (31.11)$$

provided the series on the left converges.—This is a formula very similar to (13.52).

Similarly, for $f(n)=\mu(n)$,

$$-\log(1-x) \sum_{n=1}^{\infty} \frac{\mu(n)}{n} - x = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \log(1+x\ldots+x^{n-1}). \quad \ldots \quad (31.2)$$

But the first term on the left vanishes by (11.13).

Hence

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n} \log\,(1+x+\ldots+x^{n-1}) = -x, \qquad \ldots \qquad \ldots \quad (31.21)$$

from which again follows the result (15.3) for $x \to 1$, the series on the left supposed convergent for $x=1$.

By differentiation of (31.21), or directly from (14.1), we deduce

$$\sum_{n=2}^{\infty} \frac{\mu(n)}{n} \cdot \frac{1+2x+3x^2+\ldots+(n-1).\,x^{n-2}}{1+x+x^2+\ldots+x^{n-2}+x^{n-1}} = -1. \qquad \ldots \quad (31.22)$$

This series oscillates for $x=1$.

We obtain, however, symbolically

$$\sum_{n=2}^{\infty} \frac{\mu(n)}{n} \cdot \frac{\frac{(n-1).\,n}{2}}{n} \backsim -1,$$

and hence

$$\sum_{n=2}^{\infty} \mu(n)\left(1-\frac{1}{n}\right) \backsim \sum_{n=1}^{\infty} \mu(n)\left(1-\frac{1}{n}\right) \backsim$$

$$\backsim \sum_{n=1}^{\infty} \mu(n) - \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \backsim -2,$$

from which follows

$$\sum_{n=1}^{\infty} \mu(n) \backsim -2. \quad \ldots \quad \ldots \quad \ldots \quad (31.3)$$

We may therefore regard it as highly probable, that

$$\operatorname*{Lt}_{x\to 1} \sum_{n=1}^{\infty} \mu(n).\, x^n = -2. \quad \ldots \quad \ldots \quad (31.31)$$

This agrees very well with the theorem of Frobenius. In fact, we find, putting

$$\sigma(n) = \sum_{r=1}^{n} \mu(r), \quad \ldots \quad \ldots \quad \ldots \quad (16.12)$$

for N = 83, 91, 126, 127, 128, 131, 134, 144, etc.

that $$\frac{1}{\mathrm{N}} \sum_{n=1}^{\mathrm{N}} \sigma(n) = -2$$

exactly.

VII. Conclusion.

In the present paper I have tried to show, how asymptotic formulæ in the theory of numbers may easily be obtained in a manner, somewhat different from the usual, which rests chiefly on the theory of Dirichlet's series. Incidentally I have derived many other interesting results, several of which appear to be novel. I have dealt mainly with the properties of a class of series, bearing the name of Laguerre, and closely associated with that of Dirichlet. The literature, dealing with the latter class, however, is rapidly attaining enormous proportions while that of the former is very inconsiderable, except in the special case, known as Lambert's series. Laguerre's class of series thus deserves a closer attention than hitherto has been paid to it by workers in the theory of numbers. There is reason to believe that the theory of these series might be brought to at least the same degree of perfection as that of Dirichlet's series at present. Probably new light would thus be thrown on some of the most difficult parts of the subject in particular on the still unproved, highly important, proposition of Riemann about the imaginary roots of the ζ—function. The chief usefulness of the series in question lies in the fact, that they form a connecting link between Dirichlet's series on one hand and ordinary power-series on the other, they being associated in a simple manner with both.

SHORT NOTES.

Note on Triangles inscribed in an Ellipse.*

§ 1. Consider a triangle ABC inscribed in the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\ ;$$

and let the eccentric angles of its corners be α, β, γ respectively.

If (x, y) be the centre of either of the in and ex-circles of the triangle ABC, and p_1, p_2, p_3 the perpendiculars from its vertices on the line $x=X$, we shall obviously have

$$BC.p_1 \pm CA.p_2 \pm AB.p_3=0\ .$$

But the equation of a circle having double contact with the ellipse along $x=X$ is,

$$x^2+y^2-2e^2xX+e^2X^2-b^2=0,$$

and therefore, if t_1, t_2, t_3 be the tangents from A,B,C, we have

$$t_1=ep_1\ ;\ t_2=ep_2 \text{ and } t_3=ep_3;$$

and consequently,

$$BC.\ t_1 \pm CA.\ t_2 \pm AB.\ t_3=0\ ;$$

which shows that the circumcircle of ABC touches the circle having double contact with the ellipse along the line $x=X$ parallel to its major axis. Similarly for the circle having double contact along the line $y=Y$.

Hence, if the circle ABC be $(x-h)^2+(y-k)^2=R^2$, the abscissae x_1, x_2, x_3, x_4 of the four incentres of ABC, are given as the roots of $[2h(h-e^2X)+2k^2-h^2-k^2+R^2+e^2X^2-b^2]^2=R^2\{4(h-e^2X)^2+4k^2\}$.

Hence, if p_1' p_2' p_3' and p_4' be the perpendiculars from the four incentres of ABC, upon a chord $x=d$,

$$\begin{aligned}e^4p_1'p_2'p_3'p_4' &= e^4(d-x_1)(d-x_2)(d-x_3(d-x_4)\\ &=(h^2+k^2-2e^2dh+e^2d^2+R^2-b^2)-4R^2\{(h-e^2d)^2+k^2\}\\ &=(\delta^2+R^2-r^2)-4R^2\delta^2\\ &=[\delta^2-(R-r)^2][\delta^2-(R+r)^2]=t^2t'^2,\end{aligned}$$

where δ is the distance between the circumcentre of ABC and the centre of the circle having double contact along $x=d$; and t, t' the lengths of the common tangents of these circles. From this result, it follows that if the circle round ABC were to touch the circle of double contact, since either t or t' is zero, one at least of the quantities p'_1, p'_2, p'_3, p'_4 is zero, so that the chord of contact with the ellipse must pass through the in-or excentre of the triangle.

* Extracted from a paper read by the author before the Conference of the Indian Mathematical Society, December, 1916.

Again, if $(x-X)^2+(y-Y)^2-r_1^2=0$ be one of these circles the lines through the origin parallel to the lines of intersection of this circle with the original ellipse, are.

$$\lambda\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)-(x^2+y^2)=0,$$

so that $\lambda=\rho^2$, where ρ is the radiant of the ellipse parallel to the chord of intersection. But the invariant relation between the ellipse and the circle gives

$$\lambda^3\Delta-\lambda^2\Theta+\lambda\Theta'-\Delta'=0$$

so that $\Sigma\rho_1^2=\Theta/\Delta$; $\Sigma\rho_1^2\rho_2^2=\Theta'/\Delta$ and $\rho_1^2\rho_2^2\rho_3^2=\Delta'/\Delta$

and the relation $\Theta^2=4\Delta\Theta'$ becomes $(\Sigma\rho_1)^2-4\Sigma\rho_1^2\rho_2^2=0$

or $$\rho_1\pm\rho_2\pm\rho_3=0$$

i.e. the algebraic sum of the three radiants is zero.

§ 2. The following general results, which follow as corollaries from the preceding, have been kindly suggested by Prof. A. C. L. Wilkinson.

If ABCD is a quadrilateral inscribed in a circle, the sixteen inscribed and escribed centres of the four triangles ABC, ABD, BCD, ACD lie by fours on two of perpendicular lines :

Also, $$r=\frac{1}{2R}[(x-h)^2+(y-k)^2-R^2],$$ for any incircle

Hence

$$r_1+r_2=\frac{1}{2R}[(x_1-h)^2+(y_1-k)^2-R^2+(x_2-h)^2+(y_2-k)^2-R^2]$$

$$=\frac{1}{2R}[(x_1-h)^2+(y_2-k)^2-R^2+(x_2-h)^2+(y_1-k)^2-R^2]$$

$$=r_3+r_4\,;$$

i.e. the sum of the radii of the circles inscribed in ABC and ACD is equal to the sum of the corresponding radii of ABD and BCD. We have also similar theorems for the sets of escribed circles. Again when we make

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1+\lambda(x-X)^2=0 \text{ a circle}$$

we also make $$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1+\lambda(x-X)^2+\mu\,[(x-h)^2+(y-k)^2-R^2]=0$$

a circle, and the radical axis with

$$(x-h)^2+(y-h)^2-R^2=0$$

is the same for both, so that if it touches the first it also touches the second. Hence the point on the circle ABCD where the circle having double contact with the ellipse touches the given circle is the same for all conics through ABCD. Therefore in the special case of the line-pair (as conic) we obtain the following theorem :

If $S_1, S_2 \ldots S_8$ are the points of contact of the 8 circles touching two given straight lines AC, BD and a given circle, then the same 8 points occur also as the points of contact of circles touching AB, CD and the given circles, as well as of the circles touching BC, AD and the given circle.

A. N. RAGHAVACHAR.

On poristic Polygons.

The following result (due to Salmon) furnishes a simple method of determining the condition for the porism of a polygon in terms of the invariants of its in-and circum-conics :—

If a triangle is inscribed in a conic $S=0$ *whose sides touch respectively the conics*

$$S'+lS=0,\ S'+mS=0 \text{ and } S'+nS=0,$$

then

$$\{\odot'-\triangle(mn+nl+lm)\}^2=4[\triangle'+lmn\triangle][\odot+\triangle(l+m+n)]$$

where $\odot$, $\odot'$, $\triangle$, $\triangle'$ *are the invariants of the conics* S and S'.

(*Conic Sections*, p. 343.)

For, consider a polygon $A_1A_2 \ldots A_n$ inscribed in a conic $S=0$, and such that all its sides, except A_1A_n, touch a conic S'. Then it is seen that A_1A_n touches a conic of the form $S'+nS=0$, by a repeated application of Salmon's theorem thus: put $l=m=0$, and $n=l_1$; then the side A_1A_3 of the triangle $A_1A_2A_3$ touches a conic $S'+l_1S=0$; again put $l=l_1$, $m=0$, $n=l_2$, then the side A_1A_4 of the $\triangle A_1A_3A_4$ touches $S'+l_2S=0$; and so on. Now since any side A_pA_q may be looked upon as the open side of a polygon, it follow that A_pA_q touches a conic whose equation is of the form $S'+l_{p,q}S=0$, so that if we apply Salmon's result to the two triangles $A_1A_nA_{n-1}$ and $A_1A_{n-1}A_{n-2}$, we obtain

$$[\odot'-\triangle l_{1,n-2}\, l_{1,n}]^2=4\triangle'[\odot+\triangle(l_{1,n-1}+l_{1,n})]$$

and

$$[\odot'-\triangle l_{1,n-2}\, l_{1,n-1}]^2=4\triangle'[\odot+\triangle(l_{1,n-2}+l_{1,n-1})].$$

On simplification these give the result

$$l_{1,n}=\frac{2\odot}{\triangle}\frac{1}{l_{1,n-1}}+\frac{4\triangle'}{\triangle}\frac{1}{l^2_{1,n-2}}-l_{1,n-2}$$

which may be looked upon as a recurring relation from which any $l_{1,n}$ may be determined if the two preceding ones are known.

Obviously the condition that is necessary and sufficient for the polygon to be poristic is that $l_{1,\ n}=0$.

Putting successively $n=3$, 4 in the above recurring formula, we get

$$4\triangle\triangle' l_{1,3}=\odot'^2-4\triangle'\odot,$$

and $$(\odot'^2-4\triangle'\odot)^2 l_{1,4}=8\triangle'(\odot'^3-4\triangle\odot'\odot+8\triangle'^2\triangle)$$

which give the well-known results for the case of the triangle and the quadrilateral.

This general method is obviously laborious for higher values of n, but there may exist shorter methods for special cases.

In the case of a pentagon, it is obvious that A_1A_3, A_1A_4 may be taken to touch the same conic $S'+l_{1,3}S=0$, while of course, A_3A_4 touches $S=0$. Directly applying salmon's theorem, we obtain

$$[\odot'-\triangle l_{1,3}^2]^2=4\triangle'[\odot+2\triangle l_{1,3}]$$

which gives on reduction

$$(\odot'^3-4\triangle'\odot)^2=32\triangle'^2\triangle[\theta'^3-4\triangle'\odot'\odot+8\triangle'^2\triangle].$$

Generally when n is of the form $2m+1$, it will be seen that $l_{1,\ m}$ $l_{1,\ m+1}$.

In the case of a hexagon, similarly it is obvious that the sides of the triangle $A_1A_3A_5$ all touch the same conic $S'+l_{1,3}S=0$. Hence Salmon's theorem directly gives

$$[\odot'-3\triangle l_{1,3}]^2=4[\triangle'+\triangle l_{13}'][\odot+3\triangle l_{1,3}]$$

On substituting the value of $l_{1,3}$ this will be seen to be equivalent to

$$3[\odot'^2-4\triangle'\odot]^3+16\triangle'\odot[\odot'^2-4\triangle'\odot]^2+96\triangle'^2\triangle\odot'(\odot'^2-4\triangle'\odot)+512\triangle'^4\triangle^2=0.$$

In the solution to Q. 831 (Vol. V. J.I.M.S.), Prof. Wilkinson has obtained the condition for the porism of a hexagon in this form. Generally, when n is of the form $3p$, Salmon's theorem directly gives the condition for porism in the form

$$[\odot'-3\triangle l_{1,p+1}^2]^2=4[\triangle'+\triangle l_{1,p+1}^3][\odot+3\triangle l_{1,p+1}].$$

F. H. V. Gulasekharam.

A Geometrical Problem.

Given the circumcircle and the orthocentre of a triangle to find the locus of the in- and ex-centres.

1. Let S, H denote the circumcentre and the orthocentre, and I the *in* or *ex*-centre. Then

$$SI^2=2R.NI, \qquad \ldots \qquad \ldots \quad (1)$$

where N is the mid-point of SH or *nine points centre*, and R the *given* circumradius. Thus the locus of I is defined by equation (1), and is readily seen to be a *Cartesian*.

For, the above equation may be written

$$(x^2+y^2)^2=4R^2\{(x-c)^2+y^2\} \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

S being the origin of cartesian coordinates, and N the point $(c, 0)$.

Equation (2) may again be put into the form

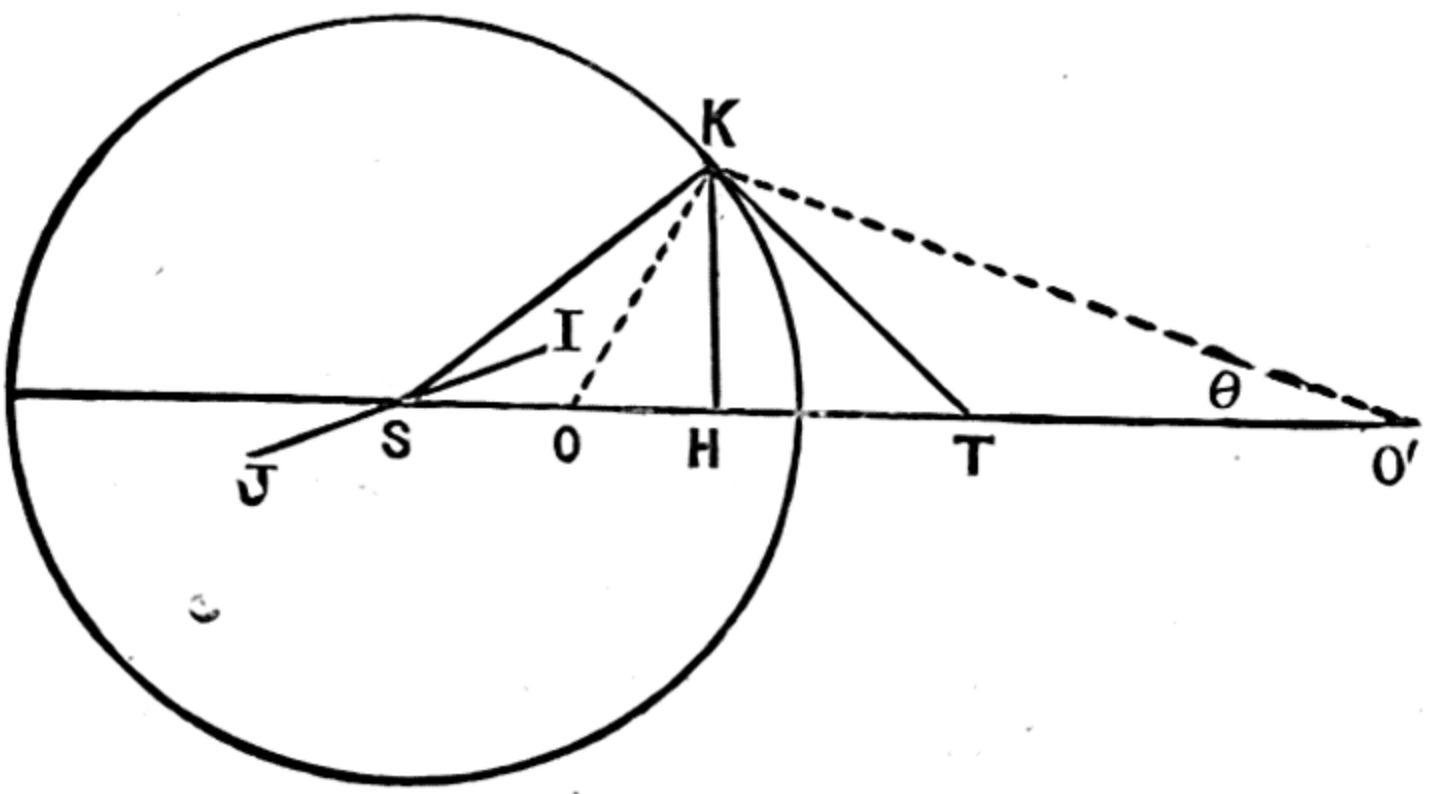
$$\{(x^2+y^2-2R^2\}^2+4R^2(2cx-R^2-c^2)=0,$$

or

$$S^2+I^2u=0, \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

where s is a circle, u a straight line and I the line at infinity. The form (3) shows that the locus is a cartesian, whose double tangent is $u=0$; the points of contact being the real points of intersection of the curve and the circle s.

2. The focal properties of the curve are found as follows :—



From T the inverse of H with respect to the circumcircle draw TK tangent to the circle. Measure TO, TO′ equal to TK. Then O, O′ are two foci of the Cartesian.

For, the circle OKO′ touches SK at K, and therefore SO/SK $=$OK/O′K$=$cot KO O′$=m$ (say)

$\therefore$

$$SO=mR.$$

and

$$SO'=R/m.$$

Also

$$SH.ST=SK^2,$$

$\therefore$

$$cR\left(m+\frac{1}{m}\right)=R^2.$$

$\therefore$

$$c=Rm/(1+m^2).$$

Denoting IO, IO′ by ρ, ρ', we have

$$\rho^2=(x-\mathrm{R}m)^2+y^2\,;$$
$$\rho'^2=(x-\mathrm{R}/m)^2+y^2.$$

$\therefore$ $$\rho^2-m^2\rho'^2=(x^2+y^2-\mathrm{R}^2)(1-m^2),$$
$$\rho^2+m^2\rho'^2=(x^2+y^2+\mathrm{R}^2)(1+m^2)-4xm\mathrm{R}.$$

$\therefore$ $$\{\rho^2-m^2\rho'^2\}^2-2k^2\{\rho^2+m^2\rho'^2\}$$
$$=(x^2+y^2-\mathrm{R}^2)^2(1-m^2)^2-2k^2\{(x^2+y^2+\mathrm{R}^2)(1+m^2)-4xm\mathrm{R}\}$$
$$=(1-m^2)^2(x^2+y^2)^2-2(x^2+y^2)\{\mathrm{R}^2(1-m^2)^2+k^2(1+m^2)\}$$
$$+8xk^2m\mathrm{R}+\mathrm{R}^4(1-m^2)^2-2k^2\mathrm{R}^2(1+m^2)$$
$$=k^4, \text{ after reduction by using (2),}$$

where $k^2=\mathrm{R}^2(1-m^2)^2/(1+m^2)=(\mathrm{SO}'-\mathrm{SO})^2/(1+1/m^2)=\mathrm{OK}^2$.

Hence, the relation between ρ and ρ' is

$$\{\rho^2-m^2\rho'^2\}^2-2k^2(\rho^2+m^2\rho'^2)+k^4=0,$$

which can be reduced to the form

$$\rho\pm m\rho'\pm k=0,$$

shewing that the foci are O, O′.

3. Now $\rho+m\rho'>m(\rho+\rho')$, since $m<1$ in the figure; and $(\rho+\rho')>\mathrm{OO}'>\mathrm{KO}'$ or k/m.

Hence $\rho+m\rho'>k$, for real values of ρ, ρ'.

The *conjugate ovals* are therefore given by $\rho-m\rho'\pm k=0$, the upper and lower signs corresponding to the inner and outer ovals.

4. The equations of the ovals in question may be written in the form

$$\rho-\rho'.\tan\theta\pm 2a\sin\theta=0\,;$$

or $$\rho\cos\theta-\rho'\sin\theta\pm a\sin 2\theta=0\,;$$

if $2a$ is written for OO′.

The fundamental property of this oval may therefore be stated as follows:—

If I is a point on the inner oval, produce IS to J such that JS=SI, and with J as centre and a radius equal to 2R describe a circle cutting the outer oval in I_1, I_2, I_3. Then the system $\mathrm{I}\,\mathrm{I}_1\,\mathrm{I}_2\,\mathrm{I}_3$ will be orthocentric, and the harmonic triangle ABC which is inscribed in the circle S will have the fixed point H for its orthocentre.

M. T. Naraniengar.

Astronomical Notes.

In Memorium.

During the last two years the Notes under this head were being written by Mr. R.J. Pocock, B.A., B.Sc, F.R.A.S., late Director of the Nizamiah Observatory, Hyderabad (Deccan), who fell a victim early last month to that fell disease, the Spanish Influenza. The deceased was much attached to his work at the Observatory, and in 1917 published a "*Star Chart*", which consists of a single circular disc which can be made to rotate so as to show at a glance the stars visible in India for any hour in the year, Very recently in the current year, H. E. H. the Nizam's Government published his "*Astrographic Catalogue.*" He wrote frequently in the *Monthly Notices of the Royal Astronomical Society*; and in India he appealed to the popular reader through his charming weekly notes on astronomical topics in the *Madras Times*. The Society regrets the loss of such an enthusiastic member; and this column, in particular, will be poorer for the lack of his contributions.

In his memory this column in the current issue has been left vacant.

SOLUTIONS.

Question 930.

(S. Malhari Rao) :—Fill up the vacant cells in the following figure with the remaining numbers of the series 1, 2, 3 ..63, 64, so that the whole figure and each of the four corner subsquares of 16 cells may all be pandiagonal magic squares. (The figure of the question is the square printed below of which only the left hand bottom corner subsquare of 16 cells was filled up. *Ed*).

Solution by R. J. Pocock, B. M. Vasavada B. Sc. and Sadanand.

[*N. B.*—There is a misprint in the question where the number 24 n the cell (6, 1) should be 34].

We may write the given quarter square in the form

$4\times8+7$	$3\times8\times4$	$4\times8\times6$	$3\times8+1$
$4\times8+2$	$3\times8+5$	$4\times8+3$	$3\times8+8$
$3\times8+3$	$4\times8+8$	$3\times8+2$	$4\times8+5$
$3\times8+6$	$4\times8+1$	$3\times8+7$	$4\times8+4$

We can construct 3 other cells in which the cofactors of 8, viz. 4 and 3 are replaced by 5, 2; 6, 1; 7, 0 respectively, we have then merely to the numbers 1, 2, 3...8 in the subsquares so that each and the whole will be a magic square. We begin by writing down the numbers 1, 2, 3, 4 arrange respectively in four superposable solutions of the eight queens problem which are consistent with the numbers already given. The remaining squares are then easily filled in, adding we obtain the solution as follows :—

$5\times8+5$	$2\times8+2$	$5\times8+\delta$	$2\times8+3$	$7\times8+1$	6	$7\times8+4$	7
$5\times8+4$	$2\times8+7$	$5\times8+1$	$2\times8+6$	$7\times8+\delta$	3	$7\times8+5$	2
$2\times8+1$	$5\times8+6$	$2\times8+4$	$5\times8+7$	5	$7\times8+2$	8	$7\times8+3$
$a\times8+\delta$	$5\times8+3$	$2\times8+5$	$5\times8+2$	4	$7\times8+7$	1	$7\times8+6$
$4\times8+7$	$3\times8+4$	$4\times8+6$	$3\times8+1$	$6\times8+3$	$8+8$	$6\times8+2$	$8+5$
$4\times8+2$	$3\times8+5$	$4\times8+3$	$3\times8+\delta$	$6\times8+6$	$8+1$	$6\times8+7$	$8+4$
$3\times8+3$	$4\times8+\delta$	$3\times8+2$	$4\times8+5$	$8+7$	$6\times8+4$	$8+6$	$6\times8+1$
$3\times8+6$	$4\times8+1$	$3\times8+7$	$4\times8+4$	$8+2$	$6\times8+5$	$8+3$	$6\times8+8$

Question 945.

(SADANAND) :—Fill up the vacant cells of the following magic square :—(The Square printed below with only the two diagonals filled)

Solution (1) *by B. M. Vasavada. B. Sc.,* (2) *by N. P. Pandya,* (3) *by E. Rama Iyer, and* (4) *by Mehta Kanhyalal.*

(1)

46	47	31	65	16
80	54	8	39	24
5	52	27	37	84
1	50	81	35	38
73	2	58	29	42

(2)

46	24	66	53	16
3	54	64	3	45
20	41	27	48	69
63	50	25	35	32
73	36	23	30	43

(3)

46	57	33	53	16
40	54	30	29	42
36	21	27	41	80
10	50	86	35	24
73	23	29	37	43

(4)

46	44	51	48	16
47	54	37	30	28
31	32	27	49	66
8	50	60	35	52
7	25	30	34	43

Question 947.

(M. K. KEWALRAMANI) :—A triangle ABC inscribed in any ellipse touches a confocal ellipse at the points D,E,F respectively. Show that the ratio of the triangle DEF to ABC is r/2R, where r and R refer to the triangle ABC.

Solution (1) *by K. B. Madhava and Hemraj,* (2) *by A. C. L. Wilkinson.*

(1) It is known that D, E and F are the points of contact of the escribed circles of ABC with its sides: cf. Smith : *Geometrical conics* P. 179 Ex 68. (Hence the actual areals of D, E, F are O, $(s-b)/a$,$(s-c)/a$; etc. We have therefore

$$\frac{\triangle DEF}{\triangle ABC}=\frac{1}{abc}\begin{vmatrix} o & s-b & s-c \\ s-a & o & s-c \\ s-a & s-b & o \end{vmatrix}$$

$$=2(s-a)(s-b)(s-c)/(abc)$$

$$=2.\frac{\triangle^2}{s.}\quad\frac{1}{4R\triangle}=\frac{r}{2R}.$$

(2) The ellipse in trilinears being $lyz+mzx+nxy=0$, we have for the tangential equation of any confocal—

$$p^2l^2+q^2m^2+r^2n^2-2plm-2qrmn-2rpnl$$
$$+\lambda(p^2+q^2+r^2-2pq\ \cos C-2qr\ \cos A-2pr\ \cos B)=0$$

This touches the sides of the triangle of reference if

$$l^2=m^2=n^2=-\lambda$$

Thus we take $yz+zx+xy=0$ for the ellipse circumscribing ABC and for the confocal

$$pq\ \sin^2\frac{C}{2}+qr\ \sin^2\frac{A}{2}+rp\ \sin^2\frac{B}{2}=0.$$

The tangential equation of the point of contact with (1, 0, 0) is

$$q\ \sin^2\frac{C}{2}+r\ \sin^2\frac{B}{2}=0.$$

Whence D is $(0, s-b, s-c)$.

It follows that $AF=CD=s-b$; $BF=CE=s-a$; $AE=BD=s-c$ and the ratio of the areas of the triangles ABC, DDF is at once seen to be $\frac{r}{2R}$.

Question 948.

(M. K. KEWALRAMANI) :—A perfectly elastic particle, acted on by on forces is projected from the centre of a rectangle whose sides are $2a$ and $2b$ $(a > b)$ to strike the bigger side first and then go on rebounding from its sides. If it ever pass through an angular point show that the direction of projection makes with the smaller side an angle whose tangent is $\frac{a}{b}.\frac{2n+1}{2m+1}$ where m and n are integers; and if it ever strike any point of the smaller side other than the extremities, the tangent of the angle must be of the form $\frac{a}{b}.\frac{2n+1}{2m}$

Solution by S. R. Ranganathan, M.A., L.T.

Let the particle first impinge on the longer side at a distance of x from its midpoint. Then the angle of projectionis $\tan^{-1}\frac{x}{b}$, so that it has to be shown that if the particle ever passes through a corner, x must be of the form $\frac{a(2n+1)}{2m+1}$.

Measuring distances parallel to the longer side from its midpoint and neglecting the *sense* of the distances, it is easily seen, because the particle is perfectly elastic, that between two consecutive impacts on a longer side, the particle is carried through a distance $2x$, irrespective of its impinging on a shorter side in the interval.

Hence, after m impactsl at the longer sides, the particle will be at a distance of $(2m+1)x$. If it ever paases through a coner, this distance must be a, or $(2n+1)a$, n being any integer. Hence, we get

$$x = \frac{a(2n+1)}{2m+1}.$$

The second part of the question does not seem to be correct. For taking the simple case of $a = \frac{3}{2}b$ and the angle of projection being $\frac{\pi}{4}$, it is easily sesn, that the second impact of the particle will be with a shorter side at a point other than an extremity. But, in such a case, according to the question $\tan\frac{\pi}{4}$ *i.e.* 1 must be of the form

$$\frac{a}{b}.\frac{2n+1}{2m} \text{ i.e. } \frac{3}{2}.\frac{2n+1}{2m}$$

which is untrue, since $3(2n+1)$ is odd and $4m$ is even.

But the condition that the angle should be of the given form is necessary that the particle may impinge at the midpoint of a shorter side.

Question 949.

(C. KRISHNAMACHARY) :—With the usual notation for the numbers of Bernoulli and Euler, show that

$$(1)\quad 2n = B_1 2^2(2^2-1)\binom{2n}{2} - B_2 2^4(2^4-1)\binom{2n}{4} + B_3 2^6(2^6-1)\binom{2n}{6} - \ldots + (-1)^{n-1} B_n 2^{2n}(2^{2n}-1)$$

$$(2)\quad (-1)^n E_n(2n+1) = (-1)^n B_n 2^{2n}(2^{2n}-1\binom{2n+1}{1} + (-1)^{n-1} B_{n-1} 2^{2n-2}(2^{2n-2}-1)\binom{2n+1}{3} \ldots + \ldots - B_1 2^2(2^2-1)\binom{2n+1}{2} + (2n+1).$$

Solution by Hemraj.

(1) This is given in the reverse after putting

$$S_{2n-1} = \frac{2^{2n}(2^{2n}-1)}{2n}B_{2n-1}$$

in Art. 575 (7) of Edwards Diff. Cal.

(2) Following the notation of Arts: 571, 572, 573, we have

$$\sin x = \cos x\left[S_1\frac{x}{1!} + S_3\frac{x^3}{!} + S^5\frac{x^5}{5!} + \ldots\ldots\right]$$

Expanding sin x and cos x, and equating the coefficients of x^{2n+1}, we get

$$S_{2n+1} = {}^{2n+1}C_2 S_{2n-1} - {}^{2n+1}C_4 S_{2n-3} + {}^{2n+1}C_6 S_{2n-5} - \ldots \ldots + (-1)^n \ldots \quad \ldots \quad \ldots \quad \text{(A)}$$

Again $\sec x + \tan x = 1 + S_1\frac{x}{1!} + S_2\frac{x^2}{2!} + S_3\frac{x^3}{3!} + \ldots = \sigma$(say)

Differenting

$$\sec x\,(\sec x + \tan x) = \frac{d\sigma}{dx} \text{ i.e. } \sec x + \tan x = \cos x\frac{d\sigma}{dx}$$

Equating the coefficients of x^{2n}, we have

$$\begin{aligned} S_{2n} &= S_{2n+1} - 2nC_2 S_{2n-1} + \ldots (-1)^n S_1 \\ &= ({}^{2n+1}C_2 - {}^{2n}C_2)S_{2n-1} - ({}^{2n+1}C_4 - {}^{2n}C_4)S_{2n-3} + ({}^{2n+1}C_6 - {}^{2n}C_6)S_{2n-5} - \ldots \\ &\quad + (-1)^{n-1}({}^{2n+1}C_{2n} - {}^{2n}C_{2n})S_1 + (-1)^n \qquad \text{from (A)} \\ &= {}^{2n}C_1 S_{2n-1} - {}^{2n}C_3 S_{2n-3} + {}^{2n}C_5 S_{2n-5} - + (-1)^n \\ &= 2^{2n}(2^{2n-1})B_{2n-1} - \frac{2n(2n-1)}{3!}B_{2n-3}2^{2n-2}(2^{2n-2}-1) + \ldots\ldots \\ &\quad + (-1)^{n-1}{}_{2n}B_1\frac{2^2(2^2-1)}{2} + (-1)^n. \end{aligned}$$

$$\therefore\ (-1)^n(2n+1)\ E_n = (-1)_n B^n 2^{2n}(2^{2n}-1)\binom{2n+1}{1} + (-1)^{n-1}B_{n-1}2^{2n-2}(2^{2n-2}-1)\binom{2n+1}{3}$$

$$+\ldots-B_1 2^2(2^2-1)^{\frac{(2n+1)}{2}}+(2n+1).$$

or $$\tan x = S_1 x + S_3 \frac{n^3}{3!} + \ldots\ldots$$

Differentiating $$\sec x = \cos x \left[S_1 + S_3 \frac{x^2}{3!} + \ldots \right]$$

$$i.e.\ 1 + S_2 \frac{x^2}{2!} + \ldots\ldots = \left[1 - \frac{x^2}{2!} + \ldots \right] \left[S_1 + S_3\ \delta \frac{x^2}{2!} + \ldots \right]$$

Equating coefficients of x^{2n}, we get the result stated.

Question 950.

(C. KRISHNAMACHARI) ;—Collect the coefficients of x^n in the series

$$\frac{1}{1-x} - \frac{x}{(1-x^2)} + \frac{1.3}{2!}\frac{x^2}{(1-x)^3} - \frac{1.3.5}{3!}\frac{x^3}{(1-x)^4} + \ldots$$

and show that their sum is zero when n is odd, and is

$$(-1)^m \cdot \frac{1.3.5\ldots(2m-1)}{2.4.6\quad\quad(2m)}$$

when n is $2m$.

Solution by Hemraj, Sadanand, Alpha, B. M. Vasavada and S. R. Ranganathan.

The given series io eqnal to

$$\frac{1}{1-x}\left\{ 1 - \tfrac{1}{2}\left(\frac{2x}{1-x}\right) + \frac{\frac{1}{2}\cdot\frac{3}{2}}{2!}\left(\frac{2x}{1-x}\right)^2 - \frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}}{3!}\left(\frac{2x}{1-x}\right)^3 + \ldots \right\}$$

$$= \frac{1}{1-x}\left\{ 1 + \frac{2x}{1-x} \right\}^{-\frac{1}{2}}$$

$$= \frac{1}{1-x}\left(\frac{1-x}{1+x}\right)^{\frac{1}{2}}$$

$$= (1-x^2)^{-\frac{1}{2}}$$

$$= 1 + \frac{1}{2}x^2 + \frac{1.3}{2,4}x^4 + \frac{1.3.5}{2.4.6}x^6 + \ldots + \frac{1.3.5.-\ldots(2m-1)}{2.4.6.\quad\quad(2m)}x^{2m} + \ldots$$

The absence of an odd power of x clearly shows that the sum of the coefficient of x^n is zero when n is odd.

The coefficient of x^{2m} is $\frac{1.3\ldots.(2m-1)}{2.4\ldots.\ (2m)}$, which shows that the sum is $\frac{1.3.5\ldots.(2m-1)}{2.4.6\ldots.\ (2m)}$ when n is equal to $2m$. The factor $(-1)^m$ with which it is multiplied in the question as printed is incorrect.

Question 951

(C. KRISHNAMACHARY):—Prove that

$$(1)\quad \int_0^{\frac{\pi}{4}} (\log \tan x)^{2n}\, dx = \frac{E_n}{2^{2n+2}}\,\pi^{2n+1},$$

$$(2)\quad \int_0^{\frac{\pi}{4}} \frac{(\log \tan x)^{2n-1}}{\cos x\,(\cos x+\sin x)}\, dx = -\frac{(2^{2n-1}-1)}{2n} B_n \pi^{2n},$$

$$(3)\quad \int_0^{\frac{\pi}{4}} \frac{(\log \tan x)^{2n-1}}{\cos 2x}\, dx = -\frac{(2^{2n}-1)}{4n} B_n \pi^{2n},$$

$$(4)\quad \int_0^{\frac{\pi}{4}} \frac{(\log \tan x)^{2n-1}}{\cos x\,(\cos x-\sin x)}\, dx = -\frac{2^{2n-1}}{2n} B_n \pi^{2n}.$$

Solution (1) *by A. C. L. Wilkinson* ; (2) *by Hemraj and Sadanand.*

For (4), we have

$$\int_0^1 \frac{(\log x)^{2n-1}}{1-x}\, dx = \sum_{r=0}^{\infty} \int_0^1 (\log x)^{2n-1} x^r dx$$

$$= -\sum_0^{\infty} \frac{(2n-1)}{(r+1)^{2n}} = -\frac{2^{2n-1} B_n}{2n} \pi^{2n}.$$

(See : Carr : *Synopsis*, It 2395—2397, whence the formula

$$B_n = 4^n \int_0^{\infty} \frac{y^{2n-1}}{e^{2\pi y}-1} dy$$

is derived ; also, Ramanujan : I. M. J., Vol III, *p.* 226.)

For (2), write x^2 in place of x and use the formula

$$\frac{2x}{1-x^2} = \frac{1}{1-x} - \frac{1}{1+x}.$$

For (3), add (2) and (4).

For (1), we repalce it by

$$\int_0^1 \frac{(\log x)^{2n}}{1+x^2}\, dx = \sum_0^{\infty} (-)\, \frac{(2n)\,!}{(2r+1)^{n+1}}.$$

In this write $x = e^{-\pi x}$ and deduce the integrals

$$\int_0^\infty \frac{\cosh 2ax}{\cosh \pi x}dx=\frac{1}{2}\sec a,\ \int_0^\infty \frac{\cos 2ax}{\cosh \pi x}dx=\frac{1}{2}\operatorname{sech} a,\left(-\frac{\pi}{2}\angle a\angle\frac{\pi}{2}\right)$$

(See Boole : *Finite Differences* p. 110).

[Compare also : Bromwich *Infinite Series*—*pp.* 452 and 473.]

(2) In each case the result follows by putting log tan $x=z$ and applying Edwards, *Diffl. Calc.* § 574.

Question 952.

(HEMRAJ):—The polar reciprocal of an equiangular spiral with respect to a rect. hyperbola having its centre at the pole and touching the spiral is the curve itself.

Solution (1) *by Sadanand*, (2) *by Alpha and L. S. Vaidyanathan, Sadanand.*

(1) Let the hyperbola and the spiral be $pr=a^2$, (2) and $r\sin\alpha=p$. Then if (R,β) be any point, its polar with respect to the hyperbola is $r\cos(\theta+\beta)=\frac{a^2}{R}$.

If it touches the spiral, then $\frac{a^2}{R}=p=r\sin\alpha$, giving $Rr=\frac{a^2}{\sin\alpha}$. The symmetry of this result shows that the polar will touch the spiral at (R,β).

Also it is clear that the curves touch where $R=r$ and hence the result.

(2) Let the rectangular hyperbola be $r^2\cos 2\theta=c^2$, and the spiral be

$$r=a\,e^{\theta\cot\alpha}.$$

These two touch at the point whose vectorial angle is $\frac{1}{2}(\pi-\alpha)$, if

$$c^2=a^2.\sin\alpha.e^{\left(\frac{\pi}{2}-\alpha\right)\cot\alpha}, \quad \ldots \quad \ldots \quad (1)$$

Now, the tangent to the spiral at (r_1, θ_1) is

$$r\cdot\cos(\theta-\overline{\theta_1-90+\alpha})=r_1\sin\alpha=a.\sin\alpha.\,e^{\theta_1\cot\alpha}.$$

i.e. $$r\cdot\sin(\alpha+\theta_1-\theta)=a\sin\alpha\ e^{\theta_1.\cot\alpha} \quad \ldots \quad \ldots \quad (2)$$

If the pole of this with respect to the rect. hyp. be (ρ,ϕ), its equation must be identical with

$$r.\rho.\cos(\theta+\phi)=c^2. \quad \ldots \quad \ldots \quad (3)$$

Hence equating coeffts of $r\cos\theta$ and $r\sin\theta$ in (2) and (3) we get

$$\frac{\sin(\alpha+\theta_1)}{\rho.\cos\phi}=\frac{\cos.(\alpha+\theta_1)}{\rho.\sin\phi}=\frac{a.e^{\theta_1\cdot\cot\alpha}.\sin\alpha}{c^2};$$

whence $\tan\phi=.\cot(\alpha+\theta_1)$, or $\phi=\frac{\pi}{2}-(\alpha+\theta_1)$;

and
$$\frac{1}{\rho}=\frac{a.e^{\theta_1\cot\alpha}\sin\alpha}{c^2}$$

$$=\frac{a\sin\alpha\, e^{\left(\frac{\pi}{2}-\alpha-\phi\right)\cot\alpha}}{a^2.\sin\alpha\, e^{\left(\frac{\pi}{2}-\alpha\right)\cot\alpha}}=\frac{e^{-\phi\cot\alpha}}{a}$$

or
$$\rho=a\,e^{\phi\cot\alpha}.$$

Hence the polar reciprocal is the locus of (ρ, ϕ) and is the original spiral itself'

Question 953.

(Communicated by Hemraj):—Two of the common tangents to a circle S and a conic T meet in P and the other two in Q. Show that P and Q lie on the same confocal to T.

Solution by A. C. L. Wilkinson.

Consider the following theorem.

S. S′ are two conics, C is a conic having double contact with S, then a conic can be described through the intersections of S′ and C which will touch a pair of common chords of S and S′ at the points where the chord of contact of the conics S, C meets them.

Let AB, CD be a pair of common chords of S and S′ and take ABC as triangle of reference; so that

$$\text{S is } yz+zx+xy=0,$$

$$\text{S}' \text{ is } ayz+bzx+cxy=0.$$

Then CD is $(a-c)\,y+(b-c)\,x=0$.

Then C is $xy+yz+zx+(\lambda x+\mu y+\nu z)^2=0$,

and consider the conic

$$xy+yz+zx+(\lambda x+\mu y+\nu z)^2-\frac{1}{c}(ayz+bzx+cxy)=0,$$

This passes through the intersections of S′ and C and since it can be written

$$z\{(c-a)'y+(c-b)\,x\}+c(\lambda x+\mu y+\nu z)^2=0,$$

it is a conic touching $z=0$, $(c-a)\,y+(c-b)\,x=0$ at the points where these lines are met by $\lambda x+\mu y+\nu z=0$.

The reciprocal theorem is

S. S′ are two conics and C a conic having double contact with S, then a conic can be described touching the four common tangents of C and S′ and passing through two of the opposite vertices of the quadrilateral formed by the four common tangents to S and S′ and the tangents at these points are the lines joining them to the pole of the chord of contact of the conics S and C with respect to S or C.

Let C degenerate to two points on S and then project these points into the circular points at infinity and the theorem as stated follows with the additional theorem that the tangents at P and Q to the confocal pass through the centre of thegiven circle. This is of course immediately obvious geometrically.

It is curious to notice that we can project this theorem into its own converse, for if we project P, Q into the circular points at infinity, the theorem to be proved becomes :

If from two points Ω, Ω′ lying on a conic C tangents be drawn to a confocal S, these tangents will touch a circle.

To prove the theorem in this form, reciprocate with respect to a focus, and we get the following theorem.

If two tangents to a circle meet another circle in four points, these four points lie on a conic having one focus at a limiting point of the two circles

This may be proved as follows :—

Take the pair of tangents

$$S\Sigma-\Delta(lx+my+n)^2=0$$

to the circle $S=0$ and consider conics through their intersection with $S'=0$, then if $S+\lambda S'=0$ be a point circle,

$$(S+\lambda S')\,\Sigma-\Delta(lx+my+n)^2=0$$

is a conic fulfilling the required conditions.

QUESTIONS FOR SOLUTION.

998. (F. H. V. GULASEKHARAM) :—If $u_1+u_2+u_3+u_4=2x$;

$$s_r=\text{sn}\,(u_r, k);\ c_r=\text{cn}(u_r, k), d_r=\text{dn}\,(u_r, k)\ ;$$

$$s'_r=\frac{\partial s_r}{\partial u_r};\ c'_r=\frac{\partial c_r}{\partial u_r};\ d'_r=\frac{\partial d_r}{\partial u_r},$$

prove that $\text{E}\,(u_1)+\text{E}(u_2)+\text{E}(u_3)+\text{E}(u_4)-2\text{E}\,(x)$

$$=\frac{k^2c_1c_2c_3c_4}{k'^2-k^2c_1c_2c_3c_4}\sum_{r=1}^{4}\frac{c_r'}{c_r}$$

$$+\frac{2k^2\,\text{sn}\,x\,\text{dn}\,x\,\text{cn}(x-u_3-u_2)}{k'^2-k^2\,\text{cn}\,x\,\text{cn}\,(x-u_3-u_2)}\frac{\text{cn}(x-u_2-u_4)\,\text{cn}\,(x-u_4-u_3)}{\text{cn}\,(x-u_2-u_4)\,\text{cn}\,(x-u_4-u_3)}$$

[*Suggested by Q.* 973.]

Prove also that the above expression is equal to the symmetrical result

$$\frac{k^2k'^2\,s_1s_2s_3s_4\sum_{r=1}^{4}\frac{s'_r}{s_r}+k^2c_1c_2c_3c_4\sum_{r=1}^{4}\frac{c'r}{cr}-d_1d_2d_3d_4\sum_{r=1}^{4}\frac{d'r}{dr}}{k'^2k^2k'^2s_1s_2s_3s_4-k^2c_1c_2c_3c_4+d_1d_2d_3d_4.}$$

999. (LAKSHMISHANKAR N. BHATT and K. J. SANJANA) :—Prove that the following function of the four quantities p, q, r, s—

$$2\Sigma p^2q^4+2\Sigma p^3q^2r-2\Sigma p^4qr-4\Sigma p^3q^3-6\Sigma p^2q^2r^2$$

is equal to $1536\ \text{cosec}^4 4\alpha$, when p, q, r, s stand for the tangents or cotangents of the angles $\alpha,\ \alpha+\frac{\pi}{4},\ \alpha+\frac{2\pi}{4},\ \alpha+\frac{3\pi}{4}$.

Prove further that the same function has the value $8\cos^2 2\alpha$, when p, q, r, s stand for the sines or cosines of the angles

$$\alpha,\ \alpha+\frac{\pi}{2},\ \alpha+\frac{2\pi}{2},\ \alpha+\frac{3\pi}{2}.$$

1000. (K. J. SANJANA) :—Prove that

$$(n-1)^{-4}+(n+1)^{-4}+(3n-1)^{-4}+(3n+1)^{-4}+(5n-1)^{-4}+(5n+1)^{-4}+\ldots$$

$$=\frac{\pi^4}{48n^4}\sec^2y\,(3\sec^2y-2),$$

$$(n-1)^{-5}-(n+1)^{-5}+(3n-1)^{-5}-(3n+1)^{-5}+(5n-1)^{-5}-(5n+1)^{-5}+\ldots$$

$$=\frac{\pi^5}{96n^5}\tan y\sec^2y\,(3\sec^2y-1),-$$

where $y\equiv\pi/2n$ and n is an integer greater than unity.

1001. (K. J. Sanjana, M.A.) :—Prove that

$$1+(2n-1)^{-4}+(2n+1)^{-4}+(4n-1)^{-4}+(4n+1)^{-4}+\ldots$$
$$=\frac{\pi^4}{48n^4}\operatorname{cosec}^2 y\,(\operatorname{cosec}^2 y-2).$$

$$1+(2n-1)^{-6}+(2n+1)^{-6}+(4n-1)^{-6}+\ldots\ldots$$
$$\frac{\pi^6}{960n^6}\operatorname{cosec}^2 y\,(15\operatorname{cosec}^4 y-15\operatorname{cosec}^2 y+2),$$

$$1+(2n-1)^{-8}+(2n+1)^{-8}+\ldots$$
$$=\frac{\pi^8\operatorname{cosec}^2 y}{80640n^8}(315\operatorname{cosec}^6 y-420\operatorname{cosec}^4 y+126\operatorname{cosec}^2 y-4),-$$

where $y\equiv\pi/2n$ and n is any positive integer.

1002. (N.P. Pandya) :—Find the lowest prime numbers which can be arranged into a magic square of nine cells. Also find a magic square of nine cells, all numbers in it being perfect squares.

1003. (S. R. Ranganathan) :—If $\phi_n(x)$ stands for the *nth.* Bernoulli's polynomial, show that

$$\phi_n\left(\frac{1}{4}\right)=2^{n-1}\left[\phi_n\left(\frac{1}{8}\right)-\phi_n\left(\frac{3}{8}\right)\right],$$

or
$$=\frac{2^{2n-2}+2^{3n-2}-2^{n-1}}{2^{2n-1}+2^{n-1}-1}\left[\phi_n\left(\frac{1}{8}\right)+\left(\frac{3}{8}\right)\right],$$

according as n, which is greater than 1, is odd or even.

1004. (C. Krishnamachari) :—Show that the effects of the elliptic motion of the earth on aberration in latitude and longitude are given respectively by

$$-C\sin\beta\,[\sin(\odot-\lambda)-e\sin(\varpi-\lambda)]$$

and
$$-C\sec\beta\,[-\cos(\odot-\lambda)+e\cos(\varpi-\lambda)]$$

where $2ra=U.T.C.\sqrt{1-e^2}\sin 1''$ the notation being as in Ball's *Spherical Astronomy.*

1005. (K. B. Madhava) :—Show that when $|x|<1$, and the real part of z is positive and less than 1, $\Gamma(1-z)=\lim_{x\to 1}(1-x)^{1-z}\,f(x,z)$, where

$$f(x,z)=\sum_{n=1}^{\infty}\cdot\frac{x^n}{n^z}.$$

1006. (K. B. Madhava) :—It is known that the hypergeometric series $F(a, b, c, x)$ diverges if $a+b=c$. Show that as $x\to 1-0$, the series however, becomes infinite like

$$\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\log\left(\frac{1}{1-x}\right).$$

1007. (Alpha) :—Prove that if

$$S_0=1!-2!+3!-4!+\ldots,$$

then $1(1!)-2(2!)+3(3!)-4(4!)+\ldots=1-2S_o$,
$1^2(1!)-2^2(2!)+3^2(3!)-4^2(4!)+\ldots=5S_o-2$;
and generally $\Sigma(-)^{n-1}n^k(n!)$ is of the form $\alpha S_o+\beta$, where α and β are integers (positive or negative).

(Bromwich : *Infinite Series*, p. 482, Ex. 21)

1008. (Communicated by Mr. Martyn M. Thomas) :—If I_n is written for the integral $\int_0^{\frac{\pi}{2}} \cos 2nx \log(2\cos\frac{1}{2}x)\,dx$,

prove that $\frac{1}{n}I_{n+1}-\frac{1}{n+1}I_n=\frac{(-)^{n+1}}{2n(n+1)(2n+1)}$;
and deduce that

$$2nI_n=\left(1-\frac{1}{3}+\frac{1}{5}-\ldots=\frac{1}{2n-1}\right)-\frac{\pi}{4}.$$

1009. (Communicated by Mr. Hemraj) :—A B C is a triangle ; O, P are its circumcentre and orthocentre. If O P meets AB, AC in B_1, C_1 ; prove that the join of the circumcentre and the orthocentre of AB_1C_1 is parallel to BC.

1010: (K. J. Sanjana, M.A.,):—Solve the equation $x^3+y^3=a^3+b^3$ where a and b are given positive quantities and x and y are required to be postive. Examples :—

$$x^3+y^3=19\equiv\left(\frac{5}{2}\right)^3+\left(\frac{3}{2}\right)^3, x^3+y^3=91=4^3+3^3.$$

1011. (Selected):—In an ellipse the tangent at P cuts the directrices in Z, Z', and the remaining tangents from Z, Z' to the ellipse meet at T. Shew that P T is normal to the ellipse and bisected by the minor axis.

1012. (N. Sankara Aiyar) :—Solve
(1) $x^6+12x^5+45x^4+100x^3+120x^2+78x+21=0$
(2) $2x^4+12x^3+30x^2+32x+12=0$
by expressing the left hand members in the forms (A^6+B^6) and (A^4+B^4), respectively.

1013. (S. R. Ranganadhan) :—If $N=a_1a_2a_3a_4$ where the a's are mutually prime integers show that the product of the numbers less than N and prime to it is given by

$$\frac{(a_1a_2a_3a_4-1)!\ \|\ (a_1a_2-1)!}{\|\left\{a_1^{(a_2-1)(a_3-1)(a_4-1)}(a_1a_2a_3-1!\ (a_1-1)!\right\}}$$

1014. (S. R. Ranganadhan):—Prove this extension of Wilson's Theorem. If m be any nonprime number, then
$\pi d(m)+(1-1)^{\frac{m-1}{2}}\equiv m$ when m iseven $\pi d(m)+(-)^{\frac{m-1}{2}}\equiv m$ when m is odd where $\pi d(m)$ denotes the product of numbers less than m and prime to it.

List of Periodicals Received.

(*From 16th September to* 15*th November* 1918).

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1. Acta Mathematica, Vol. 41, No. 3.

2. American Mathematical Monthly, Vol. 25, No. 6 June 1918.

3. Astrophyaical Journal, Vol. 47, Nos. 3 & 4, April and May 1918.

4. Bulletin of the American Mathematical Society, Vol. 25, No. 10, July 1918.

5. Mathematical Gazette, Vol. 9, No. 135, May 1918. (3 Copies).

6. Mathematical Questions and Solutions, Vol. 5, Nos. 5 & 6, and Vol. 6, Nos. 1 and 2, May, June, July, August and September 1918. (5 Copies)

7. Mathematics Teacher, Vol. 10, No. 4, June 1918.

8. Messenger of Mathematics, Vol. 47, Nos. 9 and 10, January and February 1918.

9. Monthly Notices of the Royal Astronomical Society, Vol. 78, No. 6 & 7 April and May 1918.

10. Philosophical Magazine, Vol. 36, Nos. 211 and 212, July and August 1918.

11. Proceedings of the London Mathematical Society, Vol. 17, No. 2, August 1918.

12. Proceedings of the Royal Society of London, Vol. 94, Nos. 662 and 663, June and July 1918.

13. Tohoku Mathematical Journal, Vol. 14, August 1918.

14. Transactions of the Cambridge Philosophical Society, Vol. 22, No. 1, February 1918.

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